

# Attractor Flows in $st^2$ Black Holes

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## Abstract

Following the same treatment of Bellucci et.al., we obtain, the hitherto unknown general solutions of the radial attractor flow equations for extremal black holes, both for non-BPS with non-vanishing and vanishing central charge  $Z$  for the so-called  $st^2$  model, the minimal *rank-2*  $\mathcal{N} = 2$  symmetric supergravity in  $d = 4$  space-time dimensions.

We also make useful comparisons with results that already exist in literature, and introduce the *fake supergravity (first-order) formalism* to be used in our analysis. An analysis of the *BPS bound* all along the non-BPS attractor flows and of the *marginal stability* of corresponding  $D$ -brane charge configurations has also been presented.

## I. INTRODUCTION

Black Holes [1]–[6] are truly unique objects for theoretical physicists as they pose various fascinating problems, which may offer a clue for solving the riddle of Quantum Gravity. One of the recent developments in the arena of Black Hole Physics is the issue of the *Attractor Mechanism* [7]–[10], a remarkable phenomenon occurring in case of *extremal* BHs coupled to Maxwell and scalar fields, in supersymmetric theories of gravity [11]–[74] (for further developments, see also *e.g.* [75]–[78]).

Supergravity [79] is the low-energy limit of *superstrings* [80]–[83] or *M-theory* [84–86]; in such a framework, a certain number of abelian gauge fields and moduli fields are coupled to the Einstein-Hilbert action. This is true for the theories in  $d = 4$  space-time dimensions, and having  $\mathcal{N} \geq 2$  *supercharges*, where  $4\mathcal{N}$  is the number of supersymmetries. The fermionic sector of these theories contains a certain number of spin 1/2 fermions and  $\mathcal{N}$  spin 3/2 Rarita-Schwinger fields, i.e. the *gravitinos* (the gauge fields of local supersymmetry). The vanishing of the supersymmetric variation of the gravitinos determines whether or not a certain number of supersymmetries (*BPS property*) is preserved by the BH background.

In this setting, asymptotically flat charged BH solutions, within a static and spherically symmetric *Ansatz*, mimic the famous Schwarzschild BH. A remarkable feature of electrically (and/or magnetically) charged BHs [87] as well as rotating ones [88] is a somewhat unconventional thermodynamical property called *extremality* [6, 92, 93]. *Extremal* BHs are possibly stable gravitational objects with finite entropy but vanishing temperature. *Extremality* also means that the inner (Cauchy) and outer (event) horizons do coincide, thus implying vanishing *surface gravity* (for a recent review see *e.g.* [70], and Refs. therein).

In the regime of *extremality* a particular relation among entropy, charges and spin holds, yielding that the Arnowitt-Deser-Misner (ADM) mass [89–91] is *not* an independent quantity. A beautiful phenomenon happens for Black Hole physics as the *No Hair theorem* states that there is a limited number of parameters which describe space and physical fields far away from the Black Hole. In application to the recently studied Black Holes in String theory, the parameters include mass, electric and magnetic charges and the asymptotic values of the scalar fields. These values may continuously vary, being an arbitrary point in the moduli space of the theory or, in a more geometrical language, a point in the target manifold of the scalar non-linear Lagrangian [7, 94]. It appears that for SUSY Black Holes one can prove

a stronger version of the *No Hair theorem*: Black Holes loose all their scalar hair near the horizon and their solutions in the near horizon limit are characterized only by a discrete set of parameters which correspond to conserved charges associated with the gauge symmetries. Nevertheless, the BH entropy, as given by the Bekenstein-Hawking entropy-area formula [95], is independent of the scalar charges (“*no scalar hair*”) and it only depends on the asymptotic (generally *dyonic*) BH charges in this case.

All extremal static, spherically symmetric and asymptotically flat BHs in  $d = 4$  have a *Bertotti-Robinson* [96]  $AdS_2 \times S^2$  near-horizon geometry, with vanishing scalar curvature and *conformal flatness*; in particular, the radius of  $AdS_2$  coincides with the radius of  $S^2$ , and it is proportional to the (square root of the) BH entropy (in turn proportional, through the Bekenstein-Hawking formula [95], to the area of the event horizon). Non-BPS (*i.e.* non-supersymmetric) (see *e.g.* [10, 26, 41, 43, 57, 59, 60]) extremal BHs exist as well, and they also exhibit an attractor behavior.

A particularly remarkable model in  $\mathcal{N} = 2$ ,  $d = 4$  *ungauged* supergravity is the so-called  $st^2$  model. It has been recently shown to be relevant for the special entangled quantum systems and the Freudenthal construction involving a three-qubit system consisting of one distinguished qubit and two bosonic qubits [97].

The 2 complex scalars coming from the two Abelian vector multiplets coupled to the supergravity one span the *rank-2*, completely factorized special Kähler manifold  $\frac{G}{H} = \left( \frac{SU(1,1)}{U(1)} \right)^2$ , with  $\dim_{\mathbb{C}} = 2$ ,

$$G = (SU(1,1))^2 \sim (SO(2,1))^2 \sim (SL(2, \mathbb{R}))^2 \sim (Sp(2, \mathbb{R}))^2 \quad (1.1)$$

being the  $d = 4$   $U$ -duality group<sup>1</sup>, while  $H = (U(1))^2 \sim (SO(2))^2$  is its *maximal compact subgroup*. Such a space is nothing but the element  $n = -1$  of the cubic sequence of reducible homogeneous symmetric special Kähler manifolds  $\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,2+n)}{SO(2) \otimes SO(2+n)}$  (see *e.g.* [26] and Refs. therein).

The  $st^2$  model has 1 non-BPS  $Z \neq 0$  *flat* directions, spanning the moduli space  $SO(1,1)$  (*i.e.* the scalar manifold of the  $st^2$  model in  $d = 5$ ), but *no* non-BPS  $Z = 0$  *massless Hessian modes* at all (see also [41] and [40] for a similar treatment for  $stu$  model). In other words,

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<sup>1</sup> With a slight abuse of language, we refer to  $U$ -duality group as to the *continuous* version, valid for large values of charges, of the string duality group introduced by Hull and Townsend [98].

the  $4 \times 4$  Hessian matrix of the effective BH potential at its non-BPS  $Z \neq 0$  critical points has 3 strictly positive and 1 vanishing eigenvalues (the latter correspond to *massless Hessian modes*), whereas at its non-BPS  $Z = 0$  critical points all the eigenvalues are strictly positive. After [10],  $\frac{1}{2}$ -BPS critical points of  $V_{BH}$  in  $\mathcal{N} = 2$ ,  $d = 4$  supergravity are *all* stable, and thus they determine attractors in a strict sense.

Concerning its stringy origin, the  $st^2$  model, is obtained e.g. by a  $t = u$  degeneracy of the so called  $stu$  model which can be interpreted as the low-energy limit of Type *IIA* superstrings compactified on a six-torus  $T^6$  factorized as  $T^2 \times T^2 \times T^2$ . The  $D0-D2-D4-D6$  branes wrapping the various  $T^2$ s determine the 3 magnetic and 3 electric BH charges.

The present paper studies in detail, the attractor flow equations of the  $st^2$  model, whose fundamental facts are summarized in Sect. II. In a nutshell, we reformulate all the computations done by Bellucci et. al. in [99] for  $stu$  model in the case of the much less known  $st^2$  model filling the vast gap in the existing supergravity black hole literature. All the classes of *non-degenerate* (*i.e.* with non-vanishing classical Bekenstein-Hawking [95] BH entropy) attractor flow solutions of the  $st^2$  configuration are determined, in their most general form (with all *B-fields* switched on). The main results of our investigation are listed below:

- As mentioned above, the  $\frac{1}{2}$ -BPS attractor flow solution is known since [102]-[106], and it is reviewed in Sect. III. In Sect. IV the non-BPS  $Z = 0$  attractor flow solution, untreated so far, for the  $st^2$  case is determined for the most general supporting BH charge configuration, and its relation to the supersymmetric flow, both at and away from the event horizon radius  $r_H$ , is established.
- Sect. V is devoted to the study of the non-BPS  $Z \neq 0$  attractor flow solution in full generality. By using suitable  $U$ -duality transformations (Subsect. V B 1), and starting from the  $D0 - D6$  configuration (Subsect. V A 3), the non-BPS  $Z \neq 0$  attractor flow supported by the most general  $D0 - D2 - D4 - D6$  configuration (with *all* charges switched on) is explicitly derived in Subsect. V B 2. This completes and generalizes the analyses performed in [28], [51], [59] and [62] for the  $stu$  model to the  $st^2$  case. It is also confirmed that in such a general framework: the moduli space  $SO(1, 1)$ , known to exist at the non-BPS  $Z \neq 0$  critical points of  $V_{BH}$  [41, 43], is found to be present *all along* the non-BPS attractor flow, *i.e.* for every  $r \geq r_H$ .
- In Sect. VI a detailed analysis of particular configurations, namely  $D0-D4$  (*magnetic*,

Subsect. VIA), its *dual*  $D2-D6$  (*electric*, Subsect. VIB), and  $D0-D2-D4$  (Subsect. VIC), is performed.

- The so-called *first order (fake supergravity) formalism*, introduced in [109], has been recently developed in [37] and [42] in order to describe  $d = 4$  *extremal* BHs; in general, it is based on a suitably defined real, scalar-dependent, *fake superpotential*<sup>2</sup>  $\mathcal{W}$ . In the framework of  $st^2$  model, we explicitly build up  $\mathcal{W}$  in the non-trivial cases represented by the non-BPS attractor flows. For the non-BPS  $Z = 0$  attractor flow (Sect. IV) and for non-BPS  $Z \neq 0$  attractor flow (Subsect. VB 2) *fake superpotential*  $\mathcal{W}$  is also determined.
- Within the *first-order (fake supergravity) formalism*, for all attractor flows we compute the *covariant scalar charges* as well as the *ADM mass*, studying the issue of *marginal stability* [107] for the  $st^2$  model as well. We thus complete the analysis and extend the results obtained in [51], [59] and [62] for the particular case of  $st^2$ .
- Final remarks, comments and future outlook are given in the concluding Sect. VII.

## II. BASICS OF THE $st^2$ MODEL

Cubic special Kähler geometries in  $N = 2$ ,  $d = 4$  supergravities are a subset of the special Kähler geometries describing the  $\sigma$ -model of the scalar fields in the vector multiplets. The distinguishing feature is the cubic prepotential function  $F(X^\Lambda)$ , which can arise in the large volume limit of Calabi-Yau compactifications of Type II superstrings or as reduction of minimal supergravity coupled to vector multiplets in  $d = 5$ .

Using special coordinates  $z^i = X^i/X^0 = x^i - i y^i$  ( $i = 1, \dots, n$ ), cubic special Kähler

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<sup>2</sup> It is worth pointing out that the *first order formalism*, as (re)formulated in [37] and [42] for  $d = 4$  *extremal* BHs, automatically selects the solutions which do *not* blow up at the BH event horizon. In other words, the (*covariant*) *scalar charges*  $\Sigma_i$  built in terms of the *fake superpotential*  $\mathcal{W}$  (see Eq. (3.12) further below) satisfy by construction all the conditions in order for the *Attractor Mechanism* to hold. It should be here recalled that for *extremal* BHs the solution converging at the BH event horizon ( $r \rightarrow r_H^+$ ) does *not* depend on the initial, asymptotical values of the scalar fields. See *e.g.* discussions in [33] and [44].

manifolds are described by a set of constants  $d_{ijk}$ , defining the holomorphic prepotential

$$F(X) = \frac{1}{3!} d_{ijk} \frac{X^i X^j X^k}{X^0} = (X^0)^2 f(z), \quad (2.1)$$

$$f(z) = \frac{1}{3!} d_{ijk} z^i z^j z^k. \quad (2.2)$$

The  $st^2$  model is a  $\sigma$ -model described by the coset manifold  $[\text{SU}(1, 1)/\text{U}(1)]^2$  with a cubic prepotential

$$F(X) = \frac{X^1 (X^2)^2}{X^0}, \quad (2.3)$$

which falls in the general classification given in (2.1) for  $d_{122} = 2$ . The name of the model is a consequence of the expression of the prepotential in terms of the special coordinates:

$$s = \frac{X^1}{X^0} \quad \text{and} \quad t = \frac{X^2}{X^0}, \quad (2.4)$$

which leads to  $F(X)/(X^0)^2 = f(s, t) = st^2$ .

Here we start by recalling some of the basic facts of the  $st^2$  model and hence fix our notations and conventions. The two complex moduli of the model can be defined as

$$z^1 \equiv x^1 - iy^1 \equiv s, \quad z^2 \equiv x^2 - iy^2 \equiv t \quad (2.5)$$

with  $x^i, y^i \in \mathbb{R}_0^+$  [101]. In *special coordinates* (see e.g. [100] and Refs. therein) the prepotential determining the relevant special Kähler geometry reads

$$f = st^2. \quad (2.6)$$

Working in special coordinates some of the geometric expressions take a simple and much elegant form, here for completeness we list the expressions for the Kähler potential, contravariant metric tensors, the non-vanishing component of the  $C$ -tensor, holomorphic central charge (also named superpotential) and BH effective potential ( $i = 1, 2$  throughout) :

$$\begin{aligned} K &= -\ln [-i(s - \bar{s})(t - \bar{t})^2] \Rightarrow \exp(-K) = 8y^1(y^2)^2; \\ g^{i\bar{j}} &= -\text{diag}((s - \bar{s})^2, \frac{1}{2}(t - \bar{t})^2); \\ C_{stt} &= \frac{2i}{(s - \bar{s})(t - \bar{t})^2}; \\ W(s, t) &= q_0 + q_1 s + q_2 t + p^0 st^2 - p^1 t^2 - 2p^2 st; \\ V_{BH} &= \frac{i}{2(s - \bar{s})(t - \bar{t})^2} \cdot [|W(s, t)| + |W(s, \bar{t})| + (p^1 - p^0 s)(t - \bar{t})^2] \\ &\quad [W(\bar{s}, t) + W(\bar{s}, \bar{t}) + (p^1 - p^0 \bar{s})(t - \bar{t})^2] + 2(|W(s, t)|^2 + |W(s, \bar{t})|^2). \end{aligned} \quad (2.7)$$

Thus the covariantly holomorphic Central charge function for the  $st^2$  model is (see e.g. [100] and Refs. therein)

$$\begin{aligned} Z(s, t, \bar{s}, \bar{t}) &\equiv e^{K/2} W(s, t) = \\ &= \frac{1}{\sqrt{-i(s - \bar{s})(t - \bar{t})^2}} (q_0 + q_1 s + q_2 t + p^0 s t^2 - p^1 t^2 - 2p^2 s t). \end{aligned} \quad (2.8)$$

The definition of the BH charges  $p^\Lambda$  (magnetic) and  $q_\Lambda$  (electric) ( $\Lambda = 0, 1, 2$  throughout), the effective 1-dim. (quasi-)geodesic Lagrangian of the  $st^2$  model, and the corresponding Eqs. of motion for the scalars can be computed following the same general method depicted in Subsects. 2.2 and 2.3, as well as in appendix A of [59] (treating the case  $D0 - D4$  in detail), for the general  $stu$  model and then making a degeneracy choice of  $t = u$ .

According to the Bekenstein-Hawking entropy-area formula [95], the entropy of an extremal BH in the  $st^2$  model in the Einsteinian approximation can be written as follows:

$$S_{BH} = \frac{A_H}{4} = \pi V_{BH}|_{\partial V_{BH}=0} = \pi \sqrt{|\mathcal{I}_4(\Gamma)|}, \quad (2.9)$$

where the  $(2n_V + 2) \times 1$  vector of BH charges

$$\Gamma \equiv (p^\Lambda, q_\Lambda), \quad (2.10)$$

was introduced,  $n_V$  denoting the number of Abelian vector multiplets coupled to the supergravity one (in the case under consideration  $n_V = 2$ ). Furthermore,  $\mathcal{I}_4(\Gamma)$  denotes the unique invariant of the  $U$ -duality group  $G$ , reading as follows (see e.g. Eq. (3.10) of [50], and Refs. therein):

$$\mathcal{I}_4(\Gamma) = \mathcal{I}_4, st^2(p, q) = -(p^0 q_0 + p^1 q_1)^2 + (2p^1 p^2 - p^0 q_2)(2p^2 q_0 + q_1 q_2), \quad (2.11)$$

More details regarding the relation of  $\mathcal{I}_4, st^2(p, q)$  and the so-called Cayley's Hyperdeterminant [3] can be found in Sect.5 of [50].(see also Eqn. (3.6) of [114])

In the next three Sections we will discuss the explicit solutions of the equations of motion of the scalars  $s$  and  $t$  in the dyonic background of an extremal BH of the  $st^2$  model, also named *Attractor Flow Equations*. We will consider only *non-degenerate* attractor flows, *i.e.* those flows determining a regular Black Hole solution with non-vanishing area of the horizon in the Einsteinian approximation.

As mentioned above, 3 classes of non-degenerate attractor flows exist in the  $st^2$  model:

- $\frac{1}{2}$ -BPS (Sect. III);
- non-BPS  $Z = 0$  (Sect. IV);
- non-BPS  $Z \neq 0$  (Sects. V) and (VI).

### III. THE MOST GENERAL $\frac{1}{2}$ -BPS ATTRACTOR FLOW

The explicit expression of the attractor flow solution supported by the most general  $\frac{1}{2}$ -BPS BH charge configuration in  $\mathcal{N} = 2$ ,  $d = 4$  *ungauged* supergravity coupled to  $n_V$  Abelian vector multiplets (and exhibiting a unique  $U$ -invariant  $\mathcal{I}_4$ ) is known after [102]-[106] (as well as the third of Refs. [107]):

$$\begin{aligned} \exp \left[ -4U_{\frac{1}{2}-BPS}(\tau) \right] &= \mathcal{I}_4(\mathcal{H}(\tau)); \\ z_{\frac{1}{2}-BPS}^i(\tau) &= \frac{H^i(\tau) + i\partial_{H_i}\mathcal{I}_4^{1/2}(\mathcal{H}(\tau))}{H^0(\tau) + i\partial_{H_0}\mathcal{I}_4^{1/2}(\mathcal{H}(\tau))}, \end{aligned} \quad (3.1)$$

where  $\partial_{H_i} \equiv \frac{\partial}{\partial H_i}$ , and the  $(2n_V + 2) \times 1 (= 6 \times 1$  in the model under consideration) symplectic vector

$$\mathcal{H}(\tau) \equiv (H^\Lambda(\tau), H_\Lambda(\tau)), \quad (3.2)$$

was introduced, where  $H^\Lambda(\tau)$  and  $H_\Lambda(\tau)$  are *harmonic functions* defined as follows ( $\tau \equiv (r_H - r)^{-1} \in \mathbb{R}^-$ ):

$$H^\Lambda(\tau) \equiv p_\infty^\Lambda + p^\Lambda \tau; \quad (3.3)$$

$$H_\Lambda(\tau) = q_{\Lambda, \infty} + q_\Lambda \tau,$$

such that  $\mathcal{H}(\tau)$  can be formally rewritten as

$$\mathcal{H}(\tau) = \Gamma_\infty + \Gamma \tau. \quad (3.4)$$

The asymptotical constants  $\Gamma_\infty$  must satisfy the following *integrability conditions*:

$$\mathcal{I}_4(\Gamma_\infty) = 1, \quad \langle \Gamma, \Gamma_\infty \rangle = 0, \quad (3.5)$$

where  $\langle \cdot, \cdot \rangle$  is the scalar product defined by the  $(2n_V + 2) \times (2n_V + 2) = (6 \times 6)$  symplectic metric. Under such conditions, the flow (3.1) is the most general solution of the so-called

$\frac{1}{2}$ -BPS stabilization Eqs. (see *e.g.* the recent treatment of [28]):

$$\mathcal{H}^T(\tau) = 2e^{K(z(\tau), \bar{z}(\tau))} \text{Im} \left[ W(z(\tau), \mathcal{H}(\tau)) \begin{pmatrix} \bar{X}^\Lambda(\bar{z}(\tau)) \\ \bar{F}_\Lambda(\bar{z}(\tau)) \end{pmatrix} \right], \quad (3.6)$$

obtained from the  $\frac{1}{2}$ -BPS Attractor Eqs. (see *e.g.* the treatment in [23], and Refs. therein)

$$\Gamma^T = 2e^{K(z, \bar{z})} \text{Im} \left[ W(z, \Gamma) \begin{pmatrix} \bar{X}^\Lambda(\bar{z}) \\ \bar{F}_\Lambda(\bar{z}) \end{pmatrix} \right] \quad (3.7)$$

by simply replacing  $\Gamma$  with  $\mathcal{H}(\tau)$  (see *e.g.* [103] and Refs. therein). Consistently, Eq. (3.7) is the *near-horizon* ( $\tau \rightarrow -\infty$ ) *limit* of Eq. (3.6).

Moreover, the BH charge configurations supporting the  $\frac{1}{2}$ -BPS attractors at the BH event horizon satisfy the following constraints, defining the  $\frac{1}{2}$ -BPS orbit (see Appendix II of [26] for a detailed discussion on this issue)

$$\mathcal{O}_{\frac{1}{2}\text{-BPS}} = \frac{(SU(1,1))^2}{(U(1))} \quad (3.8)$$

of the *bi-fundamental* representation  $(\mathbf{2}, \mathbf{2})$  of the  $U$ -duality group  $(SU(1,1))^2$  [26, 50]:

$$\mathcal{I}_4(\Gamma) > 0;$$

$$(p^2)^2 - p^0 q_1 \gtrless 0; \quad (3.9)$$

$$2p^1 p^2 - p^0 q_2 \gtrless 0;$$

Correspondingly,  $\mathcal{H}(\tau)$  is constrained as follows along the  $\frac{1}{2}$ -BPS attractor flow ( $\forall \tau \in \mathbb{R}^-$ ):

$$\mathcal{I}_4(\mathcal{H}(\tau)) > 0;$$

$$(H^2(\tau))^2 - H^0(\tau) H_1(\tau) \gtrless 0; \quad (3.10)$$

$$2H^1(\tau) H^2(\tau) - H^0(\tau) H_2(\tau) \gtrless 0.$$

In the *near-horizon limit*  $\tau \rightarrow -\infty$ , Eq. (3.1) yields the *purely charge-dependent, critical* expressions of the scalars at the BH event horizon. In the same limit, the constraints (3.10) consistently yield the constraints (3.9).

Consistently with the analysis of [47] performed for *stu* model, the general  $\frac{1}{2}$ -BPS attractor flow solution (3.1) of the  $st^2$  model can be *axion-free* only for the configurations  $D0 - D6$ ,  $D0 - D4$  (magnetic) and  $D2 - D6$  (electric).

As found in [108] and observed also in [59], an immediate consequence of Eq. (3.1) is that  $\Gamma_\infty$  satisfies the  $\frac{1}{2}$ -BPS *Attractor Eqs.* [103]. This determines a sort of “*Attractor Mechanism at spatial infinity*”, mapping the 4 *real* moduli  $(x^1, x^2, y^1, y^2)$  into the 6 *real* constants  $(p_\infty^1, p_\infty^2, q_{1,\infty}, q_{2,\infty})$ , arranged as  $\Gamma_\infty$  and constrained by the 2 *real* conditions (3.5).

As noticed in [59], the absence of *flat* directions in the  $\frac{1}{2}$ -BPS attractor flow (which is a general feature of  $\mathcal{N} = 2$ ,  $d = 4$  *ungauged* supergravity coupled to Abelian vector multiplets, *at least* as far as the metric of the scalar manifold is strictly positive-definite  $\forall \tau \in \mathbb{R}^-$  [10]) is crucial for the validity of the expression (3.1).

Now, by exploiting the *first-order formalism* [109] for  $d = 4$  extremal BHs [37, 42] (see also [70] and [74]), one can compute the relevant BH parameters of the  $\frac{1}{2}$ -BPS attractor flow of the  $st^2$  model starting from the expression of the  $\frac{1}{2}$ -BPS *fake superpotential*  $\mathcal{W}_{\frac{1}{2}-BPS}$ . For instance, the *ADM mass* and *covariant scalar charges* respectively read (see *e.g.* the treatments in [70] and [74]):

$$M_{ADM}(z_\infty, \bar{z}_\infty, \Gamma) = \mathcal{W}(z_\infty, \bar{z}_\infty, \Gamma) \equiv \lim_{\tau \rightarrow 0^-} \mathcal{W}(z(\tau), \bar{z}(\tau), \Gamma); \quad (3.11)$$

$$\Sigma_i(z_\infty, \bar{z}_\infty, \Gamma) = (\partial_i \mathcal{W})(z_\infty, \bar{z}_\infty, \Gamma) \equiv \lim_{\tau \rightarrow 0^-} (\partial_i \mathcal{W})(z(\tau), \bar{z}(\tau), \Gamma), \quad (3.12)$$

where the subscript “ $\infty$ ” denotes the evaluation at the moduli at spatial infinity ( $r \rightarrow \infty \Leftrightarrow \tau \rightarrow 0^-$ ). Notice that Eq. (3.11) provides, within the considered *first-order formalism*, an alternative (eventually simpler) formula for the computation of  $M_{ADM}$ , with respect to the general definition in terms of the *warp factor*  $U$  (see *e.g.* [10]):

$$M_{ADM} = \lim_{\tau \rightarrow 0^-} \frac{dU(\tau)}{d\tau}. \quad (3.13)$$

Recalling that for all  $\mathcal{N} = 2$ ,  $d = 4$  *ungauged* supergravities it holds that  $\mathcal{W}_{\frac{1}{2}-BPS} = |Z|$ , Eqs. (2.8) and (3.11) yield the following expressions of the *ADM mass* of the  $\frac{1}{2}$ -BPS attractor

flow of the  $st^2$  model:

$$\begin{aligned}
M_{ADM, \frac{1}{2}-BPS}(z_\infty, \bar{z}_\infty, \Gamma) &\equiv \lim_{\tau \rightarrow 0^-} |Z|(z(\tau), \bar{z}(\tau), \Gamma) = \\
&= \frac{|q_0 + q_1 s_\infty + q_2 t_\infty + p^0 s_\infty t_\infty^2 - p^1 t_\infty^2 - 2p^2 s_\infty t_\infty|}{\sqrt{-i(s_\infty - \bar{s}_\infty)(t_\infty - \bar{t}_\infty)^2}}.
\end{aligned} \tag{3.14}$$

Thus we have,

$$\mathcal{W}_{\frac{1}{2}-BPS} = |Z|$$

Equation (3.14) yields that the *marginal bound* [107] is not *saturated* by  $\frac{1}{2}$ -BPS states, because  $M_{ADM, \frac{1}{2}-BPS}$  is *not* equal to the sum of the *ADM masses* of the *D*-branes with appropriate fluxes (for further detail, see the discussion in [59]).

Concerning the (*covariant*) *scalar charges* of the  $\frac{1}{2}$ -BPS attractor flow of the  $st^2$  model, they can be straightforwardly computed by using Eqs. (2.8) and (3.12):

$$\begin{aligned}
\Sigma_{s, \frac{1}{2}-BPS}(z_\infty, \bar{z}_\infty, \Gamma) &\equiv \lim_{\tau \rightarrow 0^-} (\partial_s |Z|)(z(\tau), \bar{z}(\tau), \Gamma) \\
&= \lim_{\tau \rightarrow 0^-} \frac{[(\partial_s Z) \bar{Z} + Z \partial_s \bar{Z}]}{2|Z|}(z(\tau), \bar{z}(\tau), \Gamma) \\
&= \lim_{\tau \rightarrow 0^-} \frac{e^{K/2}}{2} \left[ (\partial_s K) |W| + (\partial_s W) \sqrt{\frac{\bar{W}}{W}} \right] (z(\tau), \bar{z}(\tau), \Gamma) \\
&= \frac{1}{2\sqrt{-i(s_\infty - \bar{s}_\infty)(t_\infty - \bar{t}_\infty)^2}} \cdot \\
&\quad \cdot \left[ \frac{|q_0 + q_1 s_\infty + q_2 t_\infty + p^0 s_\infty t_\infty^2 - p^1 t_\infty^2 - 2p^2 s_\infty t_\infty|}{-(s_\infty - \bar{s}_\infty)} + \right. \\
&\quad \left. + (q_1 + p^0 t_\infty^2 - 2p^2 t_\infty) \cdot \right. \\
&\quad \left. \cdot \sqrt{\frac{q_0 + q_1 \bar{s}_\infty + q_2 \bar{t}_\infty + p^0 \bar{s}_\infty \bar{t}_\infty^2 - p^1 \bar{t}_\infty^2 - 2p^2 \bar{s}_\infty \bar{t}_\infty}{q_0 + q_1 s_\infty + q_2 t_\infty + p^0 s_\infty t_\infty^2 - p^1 t_\infty^2 - 2p^2 s_\infty t_\infty}} \right].
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
\Sigma_{t, \frac{1}{2}\text{-BPS}}(z_\infty, \bar{z}_\infty, \Gamma) &\equiv \lim_{\tau \rightarrow 0^-} (\partial_t |Z|)(z(\tau), \bar{z}(\tau), \Gamma) \\
&= \lim_{\tau \rightarrow 0^-} \frac{[(\partial_t Z) \bar{Z} + Z \partial_t \bar{Z}]}{2|Z|}(z(\tau), \bar{z}(\tau), \Gamma) \\
&= \lim_{\tau \rightarrow 0^-} \frac{e^{K/2}}{2} \left[ (\partial_t K) |W| + (\partial_t W) \sqrt{\frac{\bar{W}}{W}} \right] (z(\tau), \bar{z}(\tau), \Gamma) \\
&= \frac{1}{2\sqrt{-i(s_\infty - \bar{s}_\infty)(t_\infty - \bar{t}_\infty)^2}} \cdot \\
&\quad \cdot \left[ \frac{|q_0 + q_1 s_\infty + q_2 t_\infty + p^0 s_\infty t_\infty^2 - p^1 t_\infty^2 - 2p^2 s_\infty t_\infty|}{-\frac{1}{2}(t_\infty - \bar{t}_\infty)} + \right. \\
&\quad \left. + (q_2 + 2p^0 s_\infty t_\infty - 2p^1 t_\infty - 2p^2 s_\infty) \cdot \right. \\
&\quad \left. \cdot \sqrt{\frac{q_0 + q_1 \bar{s}_\infty + q_2 \bar{t}_\infty + p^0 \bar{s}_\infty \bar{t}_\infty^2 - p^1 \bar{t}_\infty^2 - 2p^2 \bar{s}_\infty \bar{t}_\infty}{q_0 + q_1 s_\infty + q_2 t_\infty + p^0 s_\infty t_\infty^2 - p^1 t_\infty^2 - 2p^2 s_\infty t_\infty}} \right].
\end{aligned} \tag{3.16}$$

#### IV. THE MOST GENERAL NON-BPS $Z = 0$ ATTRACTOR FLOW

Let us now investigate the non-BPS  $Z = 0$  case.

As shortly noticed in [59], in spite of the fact that this attractor flow is non-supersymmetric, it has many common features with the supersymmetric ( $\frac{1}{2}$ -BPS) case.

As yielded by the analysis of [50], the non-BPS  $Z = 0$  horizon attractor solutions can be obtained from  $\frac{1}{2}$ -BPS ones simply by changing the signs of the imaginary parts of the second moduli (*dilatons*) and consistently imposing specific constraints on BH charges. Hence one has the only possible choice to flip the dilatons as follows:

$$y^1 \rightarrow y^1, \quad y^2 \rightarrow -y^2. \tag{4.1}$$

This yields the following constraints on the BH charge configurations supporting the non-

BPS  $Z = 0$  attractors at the BH event horizon ( $\tau \rightarrow -\infty$ ) [50]:

$$\mathcal{I}_4(\Gamma) > 0;$$

$$(p^2)^2 - p^0 q_1 \leq 0; \quad (4.2)$$

$$2p^1 p^2 - p^0 q_2 \geq 0.$$

The constraints (4.2) defines the non-BPS  $Z = 0$  orbit of the *bi-fundamental* representation  $(\mathbf{2}, \mathbf{2})$  of the  $U$ -duality group  $(SU(1, 1))^2$  (see Appendix II of [26])

$$\mathcal{O}_{non-BPS, Z=0} = \frac{(SU(1, 1))^2}{(U(1))}. \quad (4.3)$$

Notice that such an orbit shares the same coset expression of  $\mathcal{O}_{\frac{1}{2}-BPS}$  given by Eq. (3.8). However, they do *not* coincide, but instead they are two *separated* branches of a *disconnected* manifold, classified by the local value of the function  $sgn(|Z|^2 - |D_s Z|^2)$

The same holds *all along the attractor flow*, i.e.  $\forall \tau \in \mathbb{R}^-$ . Indeed, the most general non-BPS  $Z = 0$  attractor flow can be obtained by taking the most general  $\frac{1}{2}$ -BPS attractor flow, and flipping any one out of the two dilatons. Thus, by taking Eq. (3.1) and flipping the dilatons as given by Eq. (4.1), one achieves the following result:

$$\exp[-4U_{non-BPS, Z=0}(\tau)] = \mathcal{I}_4(\mathcal{H}(\tau));$$

$$z_{non-BPS, Z=0}^1 = \frac{H^\Lambda(\tau)H_\Lambda(\tau) - 2H^1(\tau)H_1(\tau) - i\mathcal{I}_4^{1/2}(\mathcal{H}(\tau))}{2[(H^2(\tau))^2 - H^0(\tau)H_1(\tau)]} = z_{\frac{1}{2}-BPS}^1(\tau);$$

$$z_{non-BPS, Z=0}^2 = \frac{H^\Lambda(\tau)H_\Lambda(\tau) - 2H^2(\tau)H_2(\tau) + i\mathcal{I}_4^{1/2}(\mathcal{H}(\tau))}{2[H^1(\tau)H^2(\tau) - H^0(\tau)H_2(\tau)]} = \overline{z_{\frac{1}{2}-BPS}^2(\tau)}. \quad (4.4)$$

This is the most general expression of the non-BPS  $Z = 0$  attractor flow, in the “*polarization*” given by Eq. (4.1). Consistently with the constraints (4.2),  $\mathcal{H}(\tau)$  is constrained as follows along the non-BPS  $Z = 0$  attractor flow ( $\forall \tau \in \mathbb{R}^-$ ):

$$\mathcal{I}_4(\mathcal{H}(\tau)) > 0;$$

$$(H^2(\tau))^2 - H^0(\tau)H_1(\tau) \leq 0; \quad (4.5)$$

$$2H^1(\tau)H^2(\tau) - H^0(\tau)H_2(\tau) \geq 0.$$

In the *near-horizon limit*  $\tau \rightarrow -\infty$ , Eq. (4.4) yields the *purely charge-dependent, critical* expressions of the scalars at the BH event horizon, given by Eq. (3.9) of [50]. In the same limit, the constraints (4.7) consistently yield the constraints (4.2). The *integrability conditions* (3.5) clearly hold also in this case.

Consistently with the analysis of [47], the general non-BPS  $Z = 0$  attractor flow solution (4.4) of the  $st^2$  model can be *axion-free* only for the configurations  $D0 - D6, D0 - D4$  (magnetic) and  $D2 - D6$  (electric).

A consequence of Eq. (4.4) is that  $\Gamma_\infty$  satisfies the *non-BPS  $Z = 0$  Attractor Eqs.* (see e.g. [23]). Analogously to what happens for the  $\frac{1}{2}$ -BPS attractor flow, this determines a sort of “*Attractor Mechanism at spatial infinity*”.

Analogously to what happens in the  $\frac{1}{2}$ -BPS case, the absence of *flat* directions in the non-BPS  $Z = 0$  attractor flow for the  $st^2$  model is crucial for the validity of the expression (4.4).

By exploiting the strict relation with the  $\frac{1}{2}$ -BPS attractor flow, one can also determine the explicit expression of the *fake superpotential*  $\mathcal{W}_{non-BPS, Z=0}$  for the non-BPS  $Z = 0$  attractor flow. Considering the absolute value of the  $\mathcal{N} = 2, d = 4$  *central charge function*  $Z$  given by Eq. (2.8) and flipping one out of the two dilatons in the “*polarization*” given by Eq. (4.1), one obtains the following non-BPS  $Z = 0$  *fake superpotential* (notice that  $K$ , as given by the first Eq. of (2.7), is invariant under such a flipping):

$$\begin{aligned} \mathcal{W}_{non-BPS, Z=0, s} &= e^{K/2} \left| q_0 + q_1 s + q_2 \bar{t} + p^0 \bar{t}^2 - p^1 \bar{t}^2 - 2p^2 s \bar{t} \right| = \\ &= |Z(s, \bar{t})| = \mathcal{W}_{\frac{1}{2}-BPS}(s, \bar{t}), \end{aligned} \quad (4.6)$$

where the subscript “ $s$ ” denotes the modulus untouched by the considered flipping of dilatons; in the last step we used the aforementioned fact that for all  $\mathcal{N} = 2, d = 4$  *ungauged* supergravities it holds that  $\mathcal{W}_{\frac{1}{2}-BPS} = |Z|$

Like the triality symmetry in the  $stu$  model here in the case of  $st^2$  model there is no equivalent flipping of the moduli like

$$y^1 \rightarrow -y^1, \quad y^2 \rightarrow y^2, \quad (4.7)$$

as the triality symmetry is completely broken once one chooses the last two moduli to be equal in case of  $stu$  model to generate the  $st^2$  model, and thus it is possible to have a new

symmetry for the  $st^2$  model like this:

$$y^1 \rightarrow -y^1, \quad y^2 \rightarrow e^{i\frac{\pi}{2}} y^2. \quad (4.8)$$

such that under this symmetry transformation the Kähler potential (given by the first of Eqn. 2.7) remains invariant.

Now, as shown in [99], by exploiting the *first-order formalism* [109] for  $d = 4$  extremal BHs [37, 42] (see also [70] and [74]), one can compute the relevant BH parameters of the non-BPS  $Z = 0$  attractor flow of the  $stu$  model starting from the expression of the non-BPS  $Z = 0$  *fake superpotential*  $\mathcal{W}_{non-BPS, Z=0}$ . The choice of “ $s$ -polarization”, “ $t$ -polarization” or “ $u$ -polarization” was immaterial, due to the underlying *triality symmetry* of the moduli  $s$ ,  $t$  and  $u$ . Thus, without loss of generality, they choose to perform computations in the “ $s$ -polarization” (equivalent results in the other two “*polarizations*” can be obtained by cyclic permutations of the moduli). But in our case of  $st^2$  model because of lack of triality symmetry, one can’t use the cyclic permutation of the moduli to derive results for all of them just by computing it for one. What one needs is to compute each of them separately.

The ADM mass of the non-BPS  $Z = 0$  attractor flow of the  $st^2$  model is :

$$\begin{aligned} M_{ADM, non-BPS, Z=0}(z_\infty, \bar{z}_\infty, \Gamma) &\equiv \lim_{\tau \rightarrow 0^-} \mathcal{W}_{non-BPS, Z=0, s}(z(\tau), \bar{z}(\tau), \Gamma) = \\ &= \lim_{\tau \rightarrow 0^-} |Z(s(\tau), \bar{t}(\tau))| = \\ &= \frac{\left| q_0 + q_1 s_\infty + q_2 \bar{t}_\infty + p^0 s_\infty \bar{t}_\infty^2 p^1 \bar{t}_\infty^2 - 2p^2 s_\infty \bar{t}_\infty \right|}{\sqrt{-i(s_\infty - \bar{s}_\infty)(t_\infty - \bar{t}_\infty)^2}}. \end{aligned} \quad (4.9)$$

Eq. (4.9) yields that the *marginal bound* [107] is not *saturated* by non-BPS  $Z = 0$  states, because  $M_{ADM, non-BPS, Z=0}$  is *not* equal to the sum of the *ADM masses* of the  $D$ -branes with appropriate fluxes (for further detail, see the discussion in [59]). This is actually expected, due to the strict similarity, discussed above, between  $\frac{1}{2}$ -BPS and non-BPS  $Z = 0$  attractor flows in the considered  $st^2$  model; such a similarity can be explained by noticing that both of the flows can be uplifted to the *same*  $\frac{1}{8}$ -BPS *non-degenerate* attractor flow of  $\mathcal{N} = 8$ ,  $d = 4$  supergravity (see *e.g.* the discussion in [50]).

Concerning the covariant scalar charges of the non-BPS  $Z = 0$  attractor flow of the  $st^2$  model they can be straightforwardly computed (in the “ $s$ -polarization”, and (in the “ $t$ -polarization”, separately by using Eqs. (4.6) and (3.12), but here we write the expression

for the scalar charge taking into account only the “*s-polarization*” as:

$$\begin{aligned}
\Sigma_{s,non-BPS,Z=0}(z_\infty, \bar{z}_\infty, \Gamma) &\equiv \lim_{\tau \rightarrow 0^-} (\partial_s \mathcal{W}_{non-BPS,Z=0,s})(z(\tau), \bar{z}(\tau), \Gamma) = \\
&= \lim_{\tau \rightarrow 0^-} \partial_s |Z(s(\tau), \bar{t}(\tau))| = \\
&= \lim_{\tau \rightarrow 0^-} \frac{e^{K/2}}{2} \left[ (\partial_s K) |W(s, \bar{t})| + (\partial_s W(s, \bar{t})) \sqrt{\frac{\overline{W}(\bar{s}, t)}{W(s, \bar{t})}} \right] = \\
&= \frac{1}{2\sqrt{-i(s_\infty - \bar{s}_\infty)(t_\infty - \bar{t}_\infty)^2}} \cdot \\
&\cdot \left[ \frac{|q_0 + q_1 s_\infty + q_2 \bar{t}_\infty + p^0 s_\infty \bar{t}_\infty^2 - p^1 \bar{t}_\infty^2 - 2p^2 s_\infty \bar{t}_\infty|}{-(s_\infty - \bar{s}_\infty)} + \right. \\
&\quad \left. + (q_1 + p^0 \bar{t}_\infty^2 - 2p^2 \bar{t}_\infty) \cdot \right. \\
&\quad \left. \cdot \sqrt{\frac{q_0 + q_1 \bar{s}_\infty + q_2 t_\infty + p^0 \bar{s}_\infty t_\infty^2 - p^1 t_\infty^2 - 2p^2 \bar{s}_\infty t_\infty}{q_0 + q_1 s_\infty + q_2 \bar{t}_\infty + p^0 s_\infty \bar{t}_\infty^2 - p^1 \bar{t}_\infty^2 - 2p^2 s_\infty \bar{t}_\infty}} \right] =
\end{aligned} \tag{4.10}$$

## V. THE MOST GENERAL NON-BPS $Z \neq 0$ ATTRACTOR FLOW

All the features holding for  $\frac{1}{2}$ -BPS and non-BPS  $Z = 0$  attractor flows (respectively treated in Sects. III and IV) do not directly hold for the non-BPS  $Z \neq 0$  attractor flow, which actually turns out to be rather different from (and structurally much more intricate than) the other two attractor flows.

As mentioned in the Introduction, the non-BPS  $Z \neq 0$  attractor flow of the *stu* model has been already considered in literature in particular cases, namely for the  $D0 - D4$  (*magnetic*) [51, 59],  $D0 - D6$  [59],  $D2 - D6$  (*electric*) [28, 62]  $D0 - D2 - D4$  (*magnetic* with  $D2$ ) [62],  $D0 - D2 - D4 - D6$  (without *B-fields*) [28] supporting BH charge configurations.

In the present Section we determine the explicit expression of the non-BPS  $Z \neq 0$  attractor flow for *the most general* supporting BH charge configuration, with *all* electric and magnetic charges switched on, namely for the non-BPS  $Z \neq 0$ -supporting branch of the  $D0 - D2 - D4 - D6$  configuration. Thence, as already done for  $\frac{1}{2}$ -BPS and non-BPS  $Z = 0$  attractor flows, by exploiting the *first order (fake supergravity) formalism* [37, 42, 109], we compute the *ADM masses* as well as the *covariant scalar charges*, and study the issue of

*marginal stability* [107], completing and refining the treatment given in [51, 59, 62] but for  $st^2$  model as an illustrative case.

## A. The $D0 - D6$ solution with B-fields:

### 1. $U$ -Duality Transformations along the Orbit $\mathcal{O}_{non-BPS, Z \neq 0}$

In order to derive the explicit expression of the non-BPS  $Z \neq 0$  attractor flow when all BH charges are non-vanishing, we exploit a method already used in [28], [59] and [62], based on performing suitable symplectic transformations along the relevant (*i.e.* supporting) charge orbit of the  $U$ -duality group. In Eqs. (3.8) and (4.3) we recalled the form of the  $\frac{1}{2}$ -BPS- and non-BPS  $Z = 0$ - supporting BH charge orbits of the *bi-fundamental* representation  $(\mathbf{2}, \mathbf{2})$  of the  $U$ -duality group  $G$  (given by Eq.(1.1)) of the  $st^2$  model. The corresponding non-BPS  $Z \neq 0$ -supporting BH charge orbit reads [26]

$$\mathcal{O}_{non-BPS, Z \neq 0} = \frac{(SU(1, 1))^2}{(SO(1, 1))}, \quad (5.1)$$

defined by the constraint

$$\mathcal{I}_4(\Gamma) < 0. \quad (5.2)$$

As done in [59] and [62], for the  $stu$  model, in order to perform a symplectic transformation along the charge orbit  $\mathcal{O}_{non-BPS, Z \neq 0}$  of the  $(\mathbf{2}, \mathbf{2})$  representation of the  $U$ -duality, we exploit the complete factorization of the special Kähler manifold  $\left(\frac{SU(1,1)}{U(1)}\right)^2$ , which allows one to deal with the product of two distinct  $2 \times 2$  matrices of  $SL(2, \mathbb{R})$ , rather than with a unique matrix of the  $U$ -duality group embedded in the relevant symplectic group  $Sp(6, \mathbb{R})$ .

The first step is to perform an  $Sp(6, \mathbb{R})$ -transformation from the basis  $(p^\Lambda, q_\Lambda)$  to a basis  $\mathcal{A}_{ab}$  ( $a, b = 0, 1$  throughout) of BH charges explicitly transforming under the  $(\mathbf{2}, \mathbf{2})$  of the  $U$ -duality. Such a transformation is similar to Eq. (5.1) of [59] applied for the  $stu$  case. (equivalent to Eq. (3.5) of the second Ref. of [3]; see also Section 5 of [50]). The explicit action of a generic symplectic transformation of the  $U$ -duality on the BH charges  $\mathcal{A}_{ab}$  is given by,

$$\mathcal{A}'_{a'b'} = (M_1)_{a'}^a (M_2)_{b'}^b a_{ab}; \quad (5.3)$$

$$M_i \equiv \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in SL(2, \mathbb{R}), \quad \det(M_i) = 1, \quad \forall i = 1, 2 \quad (5.4)$$

where each matrix pertains to the degrees of freedom of only one modulus (*e.g.*  $M_1$  to  $s$ ,  $M_2$  to  $t$ ). The transformation (5.3)-(5.4) of  $(SL(2, \mathbb{R}))^2 \subset Sp(6, \mathbb{R})$  induces also a *fractional linear* (*Möbius*) transformation on the moduli  $z^i$  as follows (no summation on repeated indices; also recall Eq. (2.5)):

$$z'^i = \frac{a_i z^i + b_i}{c_i z^i + d_i}. \quad (5.5)$$

As done in [59] and [62] for the *stu* model, we use the configuration  $D0 - D6$  as “*pivot*” in order to perform the transformation (5.3)-(5.5). Indeed, such a BH charge configuration supports only non-BPS  $Z \neq 0$  attractors, as it can be easily realized by computing the corresponding *quartic*  $U$ -invariant, given by Eq. (2.11):  $\mathcal{I}_4(\Gamma_{D0-D6}) < 0$ . Thus, we want to transform from the configuration  $D0 - D6$  (corresponding to charges  $(q_0, p^0)$ , which we denote here  $(q, p)$ ) to the most general configuration  $D0 - D2 - D4 - D6$ , corresponding to all BH charges switched on:  $(q_0, q_i, p^i, p^0)$ . By exploiting the transformation (5.3)-(5.5), the parameters  $\mathfrak{a}_i, \mathfrak{b}_i, \mathfrak{c}_i, \mathfrak{d}_i$  of the  $M_i$ s dualizing from  $D0 - D6$  to  $D0 - D2 - D4 - D6$  must satisfy the following set of constraints:

$$-q_0 = -a_1(a_2)^2q + b_1(b_2)^2p;$$

$$0 = -c_1(a_2)^2q + d_1(b_2)^2p;$$

$$0 = -c_2a_1a_2q + b_2d_1d_2p;$$

$$p^1 = -a_1(c_2)^2q + b_1(d_2)^2p;$$

$$p^2 = -a_2c_1c_2q + b_2d_1d_2p;$$

$$0 = -c_1(c_2)^2q + d_1(d_2)^2p; \tag{5.6}$$

Notice that the system (5.6) admits solutions iff the condition (5.2) is met; this implies the transformations (5.3)-(5.5) to belong to the  $U$ -duality orbit  $\mathcal{O}_{non-BPS,Z \neq 0}$  given by Eq. (5.1). The sign of the BH charges  $q$  and  $p$  is actually irrelevant for the condition (5.2) to be satisfied; thus, without loss of any generality, one can choose *e.g.*  $q > 0, p > 0$ . Within such a choice, the explicit form of the matrices  $M_i$ s under consideration (and of their inverse) reads as follows:

$$M_i = -\frac{1}{\sqrt{2\lambda_0\varrho_i}} \begin{pmatrix} \varrho_i\lambda_0 & -\varrho_i \\ \lambda_0 & 1 \end{pmatrix} \Leftrightarrow M_i^{-1} = -\frac{1}{\sqrt{2\lambda_0\varrho_i}} \begin{pmatrix} 1 & \varrho_i \\ -\lambda_0 & \varrho_i\lambda_0 \end{pmatrix}; \tag{5.7}$$

Now we define all parameters of the matrices  $M_i$  and their inverse:

$$\lambda \equiv \lambda_0 \left[ \frac{2p^1(p^2)^2 + p^0 \left( \sqrt{|\mathcal{I}_4|} - p^\Lambda q_\Lambda \right)}{2p^1(p^2)^2 - p^0 \left( \sqrt{|\mathcal{I}_4|} - p^\Lambda q_\Lambda \right)} \right]^{1/3} \in \mathbb{R}; \quad \lambda_0 = \left( \frac{p}{q} \right)^{\frac{1}{3}} \tag{5.8}$$

$$\varrho_i \equiv \frac{\sqrt{|\mathcal{I}_4|} - p^\Lambda q_\Lambda + 2p^i q_i}{\epsilon_{ijk} p^j p^k - 2p^0 q_i} \in \mathbb{R} \text{ (no sum on } i). \tag{5.9}$$

For the  $st^2$  model the unique quartic invariant reads as follows:

$$\mathcal{I}_4, st^2(p, q) = -(p^0 q_0 + p^1 q_1)^2 + (2p^1 p^2 - p^0 q_2)(2p^2 q_0 + q_1 q_2) \tag{5.10}$$

Thus the two values of  $\varrho$  for the  $D0 - D6$  configuration with B fields are given by:

$$\varrho_1 \equiv 2\sqrt{\frac{q_0 p^1}{(p^2)^2}}; \quad (5.11)$$

$$\varrho_2 \equiv 2\sqrt{\frac{q_0}{p^1}}; \quad (5.12)$$

## 2. $D0 - D6$ : the Most General Flow and Fake Superpotential

The most general non-BPS  $Z \neq 0$  attractor flow in the  $D0 - D6$  configuration reads as follows:

$$\exp[-4U_{non-BPS,Z \neq 0}(\tau)] = \left[ a - (-\mathcal{I}_4)^{1/4} \tau \right] \left[ k^1 - (-\mathcal{I}_4)^{1/4} \tau \right] \left[ k^2 - (-\mathcal{I}_4)^{1/4} \tau \right]^2 - b^2; \quad (5.13)$$

$$\begin{aligned} x_{non-BPS,Z \neq 0}^1(\tau) &= \lambda_0^{-1} e^{\alpha_1} \cdot \\ &\cdot \frac{\left[ k^2 - (-\mathcal{I}_4)^{1/4} \tau \right]^2 - \left[ a - (-\mathcal{I}_4)^{1/4} \tau \right] \left[ k^1 - (-\mathcal{I}_4)^{1/4} \tau \right]}{\left[ k^2 - (-\mathcal{I}_4)^{1/4} \tau \right]^2 + \left[ a - (-\mathcal{I}_4)^{1/4} \tau \right] \left[ k^1 - (-\mathcal{I}_4)^{1/4} \tau \right] - 2b}; \end{aligned} \quad (5.14)$$

$$\begin{aligned} x_{non-BPS,Z \neq 0}^2(\tau) &= \lambda_0^{-1} e^{\alpha_2} \cdot \\ &\cdot \frac{\left[ k^1 - a \right] \left[ k^2 - (-\mathcal{I}_4)^{1/4} \tau \right]}{\left[ k^1 + a - 2(-\mathcal{I}_4)^{1/4} \tau \right] \left[ k^2 - (-\mathcal{I}_4)^{1/4} \tau \right] - 2b}; \end{aligned} \quad (5.15)$$

$$\begin{aligned} y_{non-BPS,Z \neq 0}^1(\tau) &= 2\lambda_0^{-1} e^{\alpha_1} \cdot \\ &\cdot \frac{\exp[-2U_{non-BPS,Z \neq 0}(\tau)]}{\left[ k^2 - (-\mathcal{I}_4)^{1/4} \tau \right]^2 + \left[ a - (-\mathcal{I}_4)^{1/4} \tau \right] \left[ k^1 - (-\mathcal{I}_4)^{1/4} \tau \right] - 2b}; \end{aligned} \quad (5.16)$$

$$\begin{aligned} y_{non-BPS,Z \neq 0}^2(\tau) &= 2\lambda_0^{-1} e^{\alpha_2} \cdot \\ &\cdot \frac{\exp[-2U_{non-BPS,Z \neq 0}(\tau)]}{\left[ k^1 + a - 2(-\mathcal{I}_4)^{1/4} \tau \right] \left[ k^2 - (-\mathcal{I}_4)^{1/4} \tau \right] - 2b}, \end{aligned} \quad (5.17)$$

where

$$\lambda_0 \equiv (p/q)^{1/3}, \quad (5.18)$$

$a \in \mathbb{R}_0$ ,  $b, k^i \in \mathbb{R}$  ( $k^i$ 's cannot all vanish), and the doublet of real constants  $\alpha_i$  satisfies the constraint

$$\sum_{i=1,2} \alpha_i = 0. \quad (5.19)$$

It is worth pointing out that the  $D0 - D6$  configuration supports *axion-free* non-BPS  $Z \neq 0$  attractor flow(s); when considering the *near-horizon limit*, and thus the critical, charge-dependent values of the moduli, this is consistent with the analysis performed in [15, 41, 47]. An axion-free attractor flow solution of Eqs. (5.13)-(5.17) can be obtained e.g. by putting

$$k^i = a \quad \forall i = 1, 2 \quad (5.20)$$

and it reads as follows:

$$\begin{aligned} \exp[-4U_{non-BPS, Z \neq 0, axion-free}(\tau)] &= \left[ a - (-\mathcal{I}_4)^{1/4} \tau \right]^4 - b^2; \\ x_{non-BPS, Z \neq 0, axion-free}^i(\tau) &= 0; \\ y_{non-BPS, Z \neq 0, axion-free}^i(\tau) &= \lambda_0^{-1} e^{\alpha_i} \sqrt{\frac{\left[ a - (-\mathcal{I}_4)^{1/4} \tau \right]^2 + b}{\left[ a - (-\mathcal{I}_4)^{1/4} \tau \right]^2 - b}}. \end{aligned} \quad (5.21)$$

The non-BPS  $Z \neq 0$  fake superpotential of the first-order formalism can be computed to have the following form in the  $D0 - D6$  configuration:

$$\begin{aligned} \mathcal{W}_{non-BPS, Z \neq 0}(z, \bar{z}, q, p) &= \frac{1}{4} \frac{1}{\sqrt{-i(s - \bar{s})(t - \bar{t})^2}} \left[ |q^{1/3} + p^{1/3} e^{-\alpha_1} s|^2 |q^{1/3} + p^{1/3} e^{-\alpha_2} t|^2 \right] \cdot \\ &\cdot \left[ 1 + 2 \frac{(q^{2/3} - p^{2/3} e^{-2\alpha_1} |s|^2)(q^{2/3} - p^{2/3} e^{-2\alpha_2} |t|^2) - e^{-\alpha_1 - \alpha_2} q^{2/3} p^{2/3} (s - \bar{s})(t - \bar{t})}{|q^{1/3} + p^{1/3} e^{-\alpha_1} s|^2 |q^{1/3} + p^{1/3} e^{-\alpha_2} t|^2} + \right. \\ &\left. + \frac{(q^{2/3} - p^{2/3} e^{-2\alpha_2} |t|^2)^2 - e^{-2\alpha_2} q^{2/3} p^{2/3} (t - \bar{t})^2}{|q^{1/3} + p^{1/3} e^{-\alpha_2} t|^4} \right]. \end{aligned} \quad (5.22)$$

The axion-free version of such a fake superpotential reads as follows:

$$\begin{aligned} \mathcal{W}_{non-BPS, Z \neq 0, axion-free}(y, q, p) &= \frac{1}{2^3 \sqrt{2}} \frac{1}{\sqrt{y^1 y^2}} \left[ |q^{1/3} - ip^{1/3} e^{-\alpha_1} y^1| |q^{1/3} - ip^{1/3} e^{-\alpha_2} y^2|^2 \right] \cdot \\ &\cdot \left[ 1 + 2 \frac{\left[ q^{2/3} - p^{2/3} e^{-2\alpha_1} (y^1)^2 \right] \left[ q^{2/3} - p^{2/3} e^{-2\alpha_2} (y^2)^2 \right] + 4e^{-(\alpha_1 + \alpha_2)} q^{2/3} p^{2/3} y^1 y^2}{|q^{1/3} - ip^{1/3} e^{-\alpha_1} y^1|^2 |q^{1/3} - ip^{1/3} e^{-\alpha_2} y^2|^2} + \right. \\ &\left. + \frac{\left[ q^{2/3} + p^{2/3} e^{-2\alpha_2} (y^2)^2 \right]^2}{|q^{1/3} - ip^{1/3} e^{-\alpha_2} y^2|^4} \right]. \end{aligned} \quad (5.23)$$

Now, by exploiting the *first-order formalism* [109] for  $d = 4$  extremal BHs [37, 42] (see also [70] and [74]), one can compute the relevant BH parameters of the non-BPS  $Z \neq 0$  attractor flow of  $d = 4$   $st^2$  model in the  $D0 - D6$  configuration, starting from the expression of the non-BPS  $Z \neq 0$  *fake superpotential*  $\mathcal{W}_{non-BPS, Z \neq 0}$  given by Eq. (5.22).

Eqs.(5.22) and (3.11) yield, after some algebra, the following expression for the *ADM mass*:

$$\begin{aligned} M_{ADM, nBPS, Z \neq 0}(z_\infty, \bar{z}_\infty, \Gamma_{D0-D6}) &= \frac{P}{2^{5/2}} \cdot \left[ \left( \left[ (\Lambda^1)^{-1} + B^1 \right]^2 + 1 \right)^{1/2} \left[ (\Lambda^2)^{-2} + (B^2)^2 + 1 \right] \right. \\ &\left. + \left( \left[ (\Lambda^1)^{-1} - B^1 \right]^2 + 1 \right)^{1/2} \left( \left[ (\Lambda^2)^{-2} - (B^2)^2 - 1 \right]^2 - 4 (\Lambda^2)^{-2} \right)^{1/2} \right], \end{aligned} \quad (5.24)$$

where the quantities

$$\Lambda^i \equiv \lambda_0 y_\infty^i; \quad B^i \equiv \frac{x_\infty^i}{y_\infty^i}, \quad P \equiv p y_\infty^2 \sqrt{y_\infty^1}, \quad Q \equiv \frac{q}{y_\infty^2 \sqrt{y_\infty^1}} \quad (5.25)$$

were introduced, and, for simplicity's sake, the  $\alpha_i$ s were chosen all to vanish (i.e.  $\alpha_i = 0 \forall i = 1, 2$ ).  $P$  and  $Q$  are the *dressed charges*, i.e. a sort of *asymptotical redefinition* of the charges pertaining to  $D6$  and  $D0$  branes, respectively. On the other hand,  $\Lambda^i$  and  $B^i$  are usually named (*asymptotical brane*) *fluxes* and *B-fields*, respectively and the following condition is met (see Eq. (5.44) of [59] for *stu* case) :

$$\Lambda^1 \left[ 1 + (B^1)^2 \right] - (\Lambda^1)^{-1} = \Lambda^2 \left[ 1 + (B^2)^2 \right] - (\Lambda^2)^{-1}. \quad (5.26)$$

As observed in [59], Eq. (5.2) (along with the definitions (5.25)) yields the *marginal bound* [107] to be *saturated*, because  $M_{ADM, non-BPS, Z \neq 0}$  is equal to the sum of the *ADM masses* of four *D6*-branes with appropriate fluxes (for further detail, see the discussion in [59]).

Concerning the (*covariant*) *scalar charges*, they can be straightforwardly computed by recalling Eqs. (5.22) and (3.12), but their expressions are rather cumbersome. By denoting  $s \equiv x^1 - iy^1$  and  $t \equiv x^2 - iy^2$ , the covariant scalar charges of axion and dilaton in the *D0* – *D6* configuration respectively read

$$\begin{aligned} \Sigma_{x^1, non-BPS, Z \neq 0}(x_\infty^i, y_\infty^i, \Gamma_{D0-D6}) &= \frac{P x_\infty^1}{2\sqrt{2}(y_\infty^1)^2} \frac{(\Lambda^1)^{-1}}{\left(1 + [(\Lambda^1)^{-1} + B^1]^2\right)^{1/2}} \cdot \\ &\cdot \left[ \frac{(B^2)^2}{B^1} (\Lambda^1)^{-2} + \frac{1}{B^1} \left( (\Lambda^2)^{-1} - (\Lambda^1)^{-1} \right)^2 + 3B^1 \left( (B^2)^2 + 1 \right) + \Lambda^1 \left( (B^1)^2 + 1 \right) \left( (B^2)^2 + 1 \right) \right. \\ &\left. + 3(B^2)^2 (\Lambda^1)^{-1} - 2(\Lambda^2)^{-1} + 3(\Lambda^1)^{-1} \right]; \end{aligned} \quad (5.27)$$

$$\Sigma_{x^2, non-BPS, Z \neq 0}(x_\infty^i, y_\infty^i, \Gamma_{D0-D6}) = \frac{P x_\infty^2}{2\sqrt{2}(y_\infty^2)^2} \frac{[(\Lambda^1)^{-1} B^1 + (B^1)^2 + 1]}{\left(1 + [(\Lambda^1)^{-1} + B^1]^2\right)^{1/2}}; \quad (5.28)$$

$$\begin{aligned} \Sigma_{y^1, non-BPS, Z \neq 0}(x_\infty^i, y_\infty^i, \Gamma_{D0-D6}) &= -\frac{P}{4\sqrt{2}y_\infty^1} \frac{1}{\left(1 + [(\Lambda^1)^{-1} + B^1]^2\right)^{3/2}} \cdot \\ &\cdot \left[ (\Lambda^1)^{-4} (\Lambda^2)^{-2} + 3(\Lambda^1)^{-3} (\Lambda^2)^{-2} B^1 + (\Lambda^1)^{-2} (\Lambda^2)^{-1} \left[ 3(\Lambda^2)^{-1} \left( (B^1)^2 + 1 \right) - 2(\Lambda^1)^{-1} \right] + \right. \\ &\left. + 3(\Lambda^1)^{-1} B^1 \left( (B^1)^2 - 1 \right) \left( (B^2)^2 + 1 \right) + \left( (B^1)^4 - 1 \right) \left( (B^2)^2 + 1 \right) - (\Lambda^1 \Lambda^2)^{-2} \cdot \right. \\ &\left. \cdot \left( (B^1)^2 - 1 \right) + (\Lambda^1)^{-1} B^1 \left[ (\Lambda^2)^{-2} (B^1)^2 + (\Lambda^1)^{-2} \left( (B^2)^2 + 1 \right) + 3(\Lambda^2)^{-2} - 4(\Lambda^1 \Lambda^2)^{-1} \right] \right]; \end{aligned} \quad (5.29)$$

$$\begin{aligned} \Sigma_{y^2, non-BPS, Z \neq 0}(x_\infty^i, y_\infty^i, \Gamma_{D0-D6}) &= -\frac{P}{4\sqrt{2}y_\infty^2} \frac{1}{\left(1 + [(\Lambda^1)^{-1} + B^1]^2\right)^{1/2}} \cdot \\ &\cdot \left[ (\Lambda^1)^{-2} (\Lambda^2)^{-2} + (\Lambda^1)^{-1} (\Lambda^2)^{-2} B^2 + (\Lambda^1)^{-1} B^1 \left( (B^2)^2 - 1 \right) + \left( (B^1)^2 + 1 \right) \left( (B^2)^2 - 1 \right) \right]. \end{aligned} \quad (5.30)$$

### 3. $D0 - D6$ with equal $B$ -field:

The non-BPS  $Z \neq 0$  attractor flow in the  $D0 - D6$  configuration with equal  $B$  fields are given by:

$$\exp[-4U_{non-BPS,Z \neq 0}(\tau)] = \left[k - (-\mathcal{I}_4)^{1/4} \tau\right]^2 \left[h - (-\mathcal{I}_4)^{1/4} \tau\right]^2 - b^2; \quad (5.31)$$

$$x_{non-BPS,Z \neq 0}^i(\tau) = \lambda_0^{-1} e^{\alpha_i} \cdot \frac{(h - k) \left[h - (-\mathcal{I}_4)^{1/4} \tau\right]}{\left[h - (-\mathcal{I}_4)^{1/4} \tau\right] \left[h + k - 2(-\mathcal{I}_4)^{1/4} \tau\right] - 2b}; \quad (5.32)$$

$$y_{non-BPS,Z \neq 0}^i(\tau) = 2\lambda_0^{-1} e^{\alpha_i} \cdot \frac{\exp[-2U_{non-BPS,Z \neq 0}(\tau)]}{\left[h - (-\mathcal{I}_4)^{1/4} \tau\right] \left[h + k - 2(-\mathcal{I}_4)^{1/4} \tau\right] - 2b}, \quad (5.33)$$

An axion-free attractor flow solution can be obtained e.g. by putting

$$k = h \quad (5.34)$$

and it reads as follows:

$$\begin{aligned} \exp[-4U_{non-BPS,Z \neq 0, axion-free}(\tau)] &= \left[k - (-\mathcal{I}_4)^{1/4} \tau\right]^4 - b^2; \\ x_{non-BPS,Z \neq 0, axion-free}^i(\tau) &= 0; \\ y_{non-BPS,Z \neq 0, axion-free}^i(\tau) &= \lambda_0^{-1} e^{\alpha_i} \sqrt{\frac{\left[k - (-\mathcal{I}_4)^{1/4} \tau\right]^2 + b}{\left[k - (-\mathcal{I}_4)^{1/4} \tau\right]^2 - b}}. \end{aligned} \quad (5.35)$$

The non-BPS  $Z \neq 0$  fake superpotential  $\mathcal{W}_{non-BPS,Z \neq 0}(z, \bar{z}, q, p)$  has the same previous form by considering this condition:  $\lim_{\tau \rightarrow 0^-} s = \lim_{\tau \rightarrow 0^-} t$ .

In this case we have the following expression for the ADM mass:

$$M_{ADM, non-BPS, Z \neq 0}(z_\infty, \bar{z}_\infty, \Gamma_{D0-D6}) = \frac{P}{2^{7/2}} \left[ [(\Lambda)^{-1} + B]^2 + 1 \right]^{\frac{3}{2}} \cdot \left\{ 1 + 3 \frac{[(\Lambda)^{-2} - (B)^2 - 1]^2 + 4(\Lambda)^{-2}}{[[(\Lambda)^{-1} + B]^2 + 1]^2} \right\}, \quad (5.36)$$

where

$$\Lambda^1 \equiv \Lambda^2 \equiv \Lambda \quad (5.37)$$

Here covariant scalar charges of axion and dilaton for the two moduli fields  $s$  and  $t$  coincide and they respectively read as follows:

$$\Sigma_{x, non-BPS, Z \neq 0}(x_\infty, y_\infty, \Gamma_{D0-D6}) = 3\sqrt{2} P x_\infty \frac{(\Lambda^{-1}B + B^2 + 1)}{\sqrt{(\Lambda^{-1} + B)^2 + 1}}; \quad (5.38)$$

$$\Sigma_{y, non-BPS, Z \neq 0}(x_\infty, y_\infty, \Gamma_{D0-D6}) = -\frac{3P y_\infty}{\sqrt{2}} \frac{[\Lambda^{-4} + \Lambda^{-3}B + \Lambda^{-1}B(B^2 - 1) + B^4 - 1]}{\sqrt{(\Lambda^{-1} + B)^2 + 1}}. \quad (5.39)$$

## B. $D0 - D2 - D4 - D6$ :

### 1. $U$ -Duality Transformations along the Orbit $\mathcal{O}_{non-BPS, Z \neq 0}$

We want to transform from the configuration  $D0 - D6$  (corresponding to charges  $(q_0, p^0)$ , which we denote here  $(q, p)$ ) to the most general configuration  $D0 - D2 - D4 - D6$ , corresponding to all BH charges switched on:  $(q_0, q_i, p^i, p^0)$ . The parameters  $a_i, b_i, c_i, d_i$  of the  $M_i$ s dualizing from  $D0 - D6$  to  $D0 - D2 - D4 - D6$  must satisfy the following set of constraints (see for e.g. equation(5.6)):

$$-q_0 = -a_1(a_2)^2q + b_1(b_2)^2p;$$

$$q_1 = -c_1(a_2)^2q + d_1(b_2)^2p;$$

$$q_2 = -\frac{1}{2}c_2a_1a_2q + \frac{1}{2}b_2d_1d_2p;$$

$$p^1 = -a_1(c_2)^2q + b_1(d_2)^2p;$$

$$p^2 = -a_2c_1c_2q + b_2d_1d_2p;$$

$$p^0 = -c_1(c_2)^2q + d_1(d_2)^2p, \quad (5.40)$$

The explicit form of the matrices  $M_i$ s under consideration (and of their inverse) reads as follows:

$$M_i = -\frac{\text{sgn}(\lambda)}{\sqrt{(\varsigma_i + \varrho_i)}\lambda} \begin{pmatrix} \varsigma_i\lambda & -\varrho_i \\ \lambda & 1 \end{pmatrix} \Leftrightarrow M_i^{-1} = -\frac{\text{sgn}(\lambda)}{\sqrt{(\varsigma_i + \varrho_i)}\lambda} \begin{pmatrix} 1 & \varrho_i \\ -\lambda & \varsigma_i\lambda \end{pmatrix}; \quad (5.41)$$

$$\lambda \equiv \left(\frac{p}{q}\right)^{1/3} \left[ \frac{2p^1(p^2)^2 + p^0(\sqrt{-\mathcal{I}_4} - p^\Lambda q_\Lambda)}{2p^1(p^2)^2 - p^0(\sqrt{-\mathcal{I}_4} - p^\Lambda q_\Lambda)} \right]^{1/3} \in \mathbb{R}; \quad (5.42)$$

$$\varsigma_1 \equiv \frac{\sqrt{-\mathcal{I}_4} + p^0q_0 - p^1q_1 + p^2q_2}{2(p^2)^2 - 2p^0q_1}; \quad (5.43)$$

$$\varsigma_2 \equiv \frac{\sqrt{-\mathcal{I}_4} + p^0q_0 + p^1q_1}{2p^1p^2 - p^0q_2}; \quad (5.44)$$

$$\varrho_1 \equiv \frac{\sqrt{-\mathcal{I}_4} - p^0 q_0 + p^1 q_1 - p^2 q_2}{2(p^2)^2 - 2p^0 q_1}; \quad (5.45)$$

$$\varrho_2 \equiv \frac{\sqrt{-\mathcal{I}_4} - p^0 q_0 - p^1 q_1}{2p^1 p^2 - p^0 q_2}. \quad (5.46)$$

The above definitions (5.43)-(5.46) together with (2.11) imply that

$$\varsigma_1 \varrho_1 = -\frac{(q_2)^2 + 4q_0 p^1}{4(p^2)^2 - 4p^0 q_1}; \quad (5.47)$$

$$\varsigma_2 \varrho_2 = -\frac{q_1 q_2 + 2q_0 p^2}{2p^1 p^2 - p^0 q_2}. \quad (5.48)$$

The  $U$ -duality transformation (5.3)-(5.5) belonging to the orbit  $\mathcal{O}_{non-BPS, Z \neq 0}$ , leaves  $\mathcal{I}_4$  unchanged:

$$\mathcal{I}_4, st^2(p, q) = -(p^0 q_0 + p^1 q_1)^2 + (2p^1 p^2 - p^0 q_2)(2p^2 q_0 + q_1 q_2) = -(pq)^2 = \mathcal{I}_4(\Gamma_{D0-D6}). \quad (5.49)$$

## 2. $D0 - D2 - D4 - D6$ : the Most General Flow and Fake Superpotential

Now, by performing the  $U$ -duality transformation (5.3)-(5.5) (along with Eqs. (5.41)-(5.48)) and using the most general non-BPS  $Z \neq 0$  attractor flow in the  $D0 - D6$  configuration given by Eqs. (5.13)-(5.17), it is a matter of long but straightforward computations to determine the most general non-BPS  $Z \neq 0$  attractor flow in the most general configuration, namely in the  $D0 - D2 - D4 - D6$  one, in which *all* BH charges are switched on. It reads as follows <sup>3</sup> (the moduli are here denoted as  $\mathcal{Z}^i \equiv \mathcal{X}^i - i\mathcal{Y}^i$ ;  $i \neq j \neq l$  and no sum on repeated  $i$ -indices throughout):

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<sup>3</sup> In the particular case in which  $b = 0$ , the expression of  $\exp[-4U_{non-BPS, Z \neq 0}(\tau)]$  can be recast in the form

$$\exp[-4U_{non-BPS, Z \neq 0}(\tau)] = -\mathcal{I}_4(\mathcal{H}(\tau)),$$

consistently with the results of [28] and [59], and on the same ground of (the first of) Eqs. (3.1) and (4.4), respectively holding for the  $\frac{1}{2}$ -BPS and non-BPS  $Z = 0$  attractor flows.

$$\exp[-4U_{non-BPS,Z \neq 0}(\tau)] = h_0(\tau) h_1(\tau) h_2^2(\tau) - b^2; \quad (5.50)$$

$$\mathcal{X}_{non-BPS,Z \neq 0}^1(\tau) = \frac{\left\{ \begin{aligned} &\varsigma_1 e^{2\alpha_1} \nu^2 [h_2^2(\tau) + h_0(\tau) h_1(\tau) + 2b] + \\ &+ e^{\alpha_1} \nu (\varsigma_1 - \varrho_1) [h_2^2(\tau) - h_0(\tau) h_1(\tau)] + \\ &- \varrho_1 [h_2^2(\tau) + h_0(\tau) h_1(\tau) - 2b] \end{aligned} \right\}}{\left\{ \begin{aligned} &e^{2\alpha_1} \nu^2 [h_2^2(\tau) + h_0(\tau) h_1(\tau) + 2b] + \\ &+ 2e^{\alpha_1} \nu [h_2^2(\tau) - h_0(\tau) h_1(\tau)] + \\ &+ h_2^2(\tau) + h_0(\tau) h_1(\tau) - 2b \end{aligned} \right\}}; \quad (5.51)$$

$$\mathcal{Y}_{non-BPS,Z \neq 0}^1(\tau) = \frac{2\nu e^{\alpha_1} (\varsigma_1 + \varrho_1) \exp[-2U_{non-BPS,Z \neq 0}(\tau)]}{\left\{ \begin{aligned} &e^{2\alpha_1} \nu^2 [h_2^2(\tau) + h_0(\tau) h_1(\tau) + 2b] + \\ &+ 2e^{\alpha_1} \nu [h_2^2(\tau) - h_0(\tau) h_1(\tau)] + \\ &+ h_2^2(\tau) + h_0(\tau) h_1(\tau) - 2b \end{aligned} \right\}}, \quad (5.52)$$

$$\mathcal{X}_{non-BPS,Z \neq 0}^2(\tau) = \frac{\left\{ \begin{aligned} &\varsigma_2 e^{2\alpha_2} \nu^2 [h_1(\tau) h_2(\tau) + h_0(\tau) h_2(\tau) + 2b] + \\ &+ e^{\alpha_2} \nu (\varsigma_2 - \varrho_2) [h_1(\tau) h_2(\tau) - h_0(\tau) h_2(\tau)] + \\ &- \varrho_2 [h_1(\tau) h_2(\tau) + h_0(\tau) h_2(\tau) - 2b] \end{aligned} \right\}}{\left\{ \begin{aligned} &e^{2\alpha_2} \nu^2 [h_1(\tau) h_2(\tau) + h_0(\tau) h_2(\tau) + 2b] + \\ &+ 2e^{\alpha_2} \nu [h_1(\tau) h_2(\tau) - h_0(\tau) h_2(\tau)] + \\ &+ h_1(\tau) h_2(\tau) + h_0(\tau) h_2(\tau) - 2b \end{aligned} \right\}}; \quad (5.53)$$

$$\mathcal{Y}_{non-BPS,Z \neq 0}^2(\tau) = \frac{2\nu e^{\alpha_2}(\varsigma_2 + \varrho_2) \exp[-2U_{non-BPS,Z \neq 0}(\tau)]}{\left\{ \begin{aligned} &e^{2\alpha_2}\nu^2 [h_1(\tau)h_2(\tau) + h_0(\tau)h_2(\tau) + 2b] + \\ &+ 2e^{\alpha_2}\nu [h_1(\tau)h_2(\tau) - h_0(\tau)h_2(\tau)] + \\ &+ h_1(\tau)h_2(\tau) + h_0(\tau)h_2(\tau) - 2b \end{aligned} \right\}}, \quad (5.54)$$

where  $\varsigma_i$  and  $\varrho_i$  have been defined in Eqs. (5.43) - (5.46), respectively. The constants  $\alpha_i$  and  $b$  have been introduced in Eqs. (5.13) - (5.17). Furthermore, the new quantities (see Eqs. (5.8), (5.18) and (5.42) as well)

$$\nu \equiv \frac{\lambda}{\lambda_0} = \left[ \frac{2p^1(p^2)^2 + p^0(\sqrt{-\mathcal{I}_4} - p^\Lambda q_\Lambda)}{2p^1(p^2)^2 - p^0(\sqrt{-\mathcal{I}_4} - p^\Lambda q_\Lambda)} \right]^{1/3} \in \mathbb{R}; \quad (5.55)$$

$$h_\Lambda(\tau) \equiv b_\Lambda - (-\mathcal{I}_4)^{1/4} \tau, \quad (5.56)$$

where  $b_\Sigma$  are real constants, have been defined.

By performing the *near-horizon* (i.e.  $\tau \rightarrow -\infty$ ) *limit*, Eqs. (5.51) - (5.54), respectively yield the following *critical* values of the moduli (the subscript “ $H$ ” stands for “horizon”):

$$\mathcal{X}_{non-BPS,Z \neq 0,H}^i \equiv \lim_{\tau \rightarrow -\infty} \mathcal{X}_{non-BPS,Z \neq 0}^i(\tau) = \frac{\varsigma_i e^{2\alpha_i} \nu^2 - \varrho_i}{e^{2\alpha_i} \nu^2 + 1}; \quad \forall i = 1, 2. \quad (5.57)$$

$$\mathcal{Y}_{nBPS,Z \neq 0,H}^1 \equiv \lim_{\tau \rightarrow -\infty} \mathcal{Y}_{nBPS,Z \neq 0}^1(\tau) = \frac{1}{2} \frac{e^{\alpha_1}(\varsigma_1 + \varrho_1)\nu}{e^{2\alpha_1}\nu^2 + 1} = \frac{\sqrt{-\mathcal{I}_4} e^{\alpha_1} \nu}{(2(p^2)^2 - 2p^0 q_1)(e^{2\alpha_1}\nu^2 + 1)}, \quad (5.58)$$

$$\mathcal{Y}_{nBPS,Z \neq 0,H}^2 \equiv \lim_{\tau \rightarrow -\infty} \mathcal{Y}_{nBPS,Z \neq 0}^2(\tau) = \frac{1}{2} \frac{e^{\alpha_2}(\varsigma_2 + \varrho_2)\nu}{e^{2\alpha_2}\nu^2 + 1} = \frac{\sqrt{-\mathcal{I}_4} e^{\alpha_2} \nu}{(2p^1 p^2 - p^0 q_2)(e^{2\alpha_2}\nu^2 + 1)}. \quad (5.59)$$

It is worth pointing out that the  $D0 - D2 - D4 - D6$  configuration does *not* support *axion-free* non-BPS  $Z \neq 0$  attractor flow(s); when considering the *near-horizon limit*, and thus the critical, charge-dependent values of the moduli, this is consistent with the analysis performed in [15, 41, 47]. The solution (5.50)-(5.54) (along with the definitions (5.55) and (5.56)) generalizes the result of [28] for the particular case of  $st^2$ . As mentioned in Sect. 1, of [28], it was shown that, within the (non-BPS  $Z \neq 0$ -supporting branches of the)  $D2 - D6$

(*electric*) and  $D0 - D2 - D4 - D6$  configurations, in absence of (some of the)  $B$ -fields the attractor flow solution can be obtained by replacing the  $Sp(6, \mathbb{R})$ -covariant vector  $\Gamma$  of charges (defined by Eq. (2.10)) with the  $Sp(6, \mathbb{R})$ -covariant vector  $\mathcal{H}(\tau)$  of harmonic functions (defined by Eqs. (3.2)-(3.5)) in the corresponding critical, horizon solution.

For the  $\frac{1}{2}$ -BPS and non-BPS  $Z = 0$  attractor flows, respectively treated in Sects. III and IV, such a procedure allows one to determine the most general attractor flow solution starting from the corresponding critical, horizon solution.

On the other hand, for the non-BPS  $Z \neq 0$  attractor flow such a procedure fails in presence of non-vanishing  $B$ -fields. In other words, it can be shown that the completely general non-BPS  $Z \neq 0$  attractor flow solution (5.50)-(5.54) is *not* a solution of the *would-be non-BPS  $Z \neq 0$  stabilization Eqs.* (see the treatments of [23], [19] and [32] for *stu* BHs).

The issue concerning whether (in all non-BPS  $Z \neq 0$ -supporting configurations) the *actual non-BPS  $Z \neq 0$  stabilization Eqs.* (if any) admit a ( $\frac{1}{2}$ -)BPS-like reformulation in terms of a non-BPS  $Z \neq 0$  *fake superpotential* (whose general form is given by Eq. (5.60) below) is open, and its investigation is left for future work.

Next, we can compute the non-BPS  $Z \neq 0$  *fake superpotential* of the *first-order formalism* in the  $D0 - D2 - D4 - D6$  configuration. To do this, we apply the  $U$ -duality transformation (5.3)-(5.5) (along with Eqs. (5.41)-(5.46)) to the expression of the non-BPS  $Z \neq 0$  *fake superpotential* in the  $D0 - D6$  configuration given by Eq. (5.22), and, by noticing that  $\mathcal{W}$  does *not* have any further covariance property under such a transformation, after some algebra one achieves the following result:

$$\begin{aligned}
\mathcal{W}_{non-BPS, Z \neq 0}(s, \bar{s}, t, \bar{t}, p^0, p^1, p^2, q_0, q_1, q_2) = \\
= \frac{1}{4} \frac{\nu^{3/2} (-\mathcal{I}_4)^{1/4}}{\sqrt{(\varsigma_1 + \varrho_1)(\varsigma_2 + \varrho_2)}} e^{K/2} \left[ \left| \varsigma_1 - s + (\varrho_1 + s)e^{-\alpha_1} \nu^{-1} \right| \left| \varsigma_2 - t + (\varrho_2 + t)e^{-\alpha_2} \nu^{-1} \right|^2 \right] \cdot \\
\cdot \left( 1 + 2 \frac{[|\varsigma_1 - s|^2 - |\varrho_1 + s|^2 e^{-2\alpha_1} \nu^{-2}] [|\varsigma_2 - t|^2 - |\varrho_2 + t|^2 e^{-2\alpha_2} \nu^{-2}]}{|\varsigma_1 - s + (\varrho_1 + s)e^{-\alpha_1} \nu^{-1}|^2 |\varsigma_2 - t + (\varrho_2 + t)e^{-\alpha_2} \nu^{-1}|^2} + \right. \\
- 2 \frac{e^{-(\alpha_1 + \alpha_2)} \nu^{-2} (\varsigma_1 + \varrho_1)(\varsigma_2 + \varrho_2)(s - \bar{s})(t - \bar{t})}{|\varsigma_1 - s + (\varrho_1 + s)e^{-\alpha_1} \nu^{-1}|^2 |\varsigma_2 - t + (\varrho_2 + t)e^{-\alpha_2} \nu^{-1}|^2} + \\
\left. + \frac{[|\varsigma_2 - t|^2 - |\varrho_2 + t|^2 e^{-2\alpha_2} \nu^{-2}]^2}{|\varsigma_2 - t + (\varrho_2 + t)e^{-\alpha_2} \nu^{-1}|^4} - \frac{e^{-2\alpha_2} \nu^{-2} (\varsigma_2 + \varrho_2)^2 (t - \bar{t})^2}{|\varsigma_2 - t + (\varrho_2 + t)e^{-\alpha_2} \nu^{-1}|^4} \right). \tag{5.60}
\end{aligned}$$

Consistently with the *first-order formalism* [109] for  $d = 4$  extremal BHs [37, 42] (see also [70] and [74]), it is easy to check that the *near-horizon limit* of  $\mathcal{W}_{non-BPS,Z \neq 0}^2$  yields the square root of  $-\mathcal{I}_4$  (given by Eq. (2.11)), or equivalently the square root of the *Cayley's hyperdeterminant*  $Det(\Psi)$ :

$$\begin{aligned} & \mathcal{W}_{non-BPS,Z \neq 0,H}^2(\Gamma_{D0-D2-D4-D6}) \equiv \\ & \equiv \lim_{\tau \rightarrow -\infty} \mathcal{W}_{non-BPS,Z \neq 0}^2(\mathcal{Z}(\tau), \overline{\mathcal{Z}}(\tau), \Gamma_{D0-D2-D4-D6}) = \\ & = \sqrt{-\mathcal{I}_4} = \sqrt{Det(\Psi)} = \frac{S_{BH,non-BPZ,Z \neq 0}(\Gamma_{D0-D2-D4-D6})}{\pi}, \end{aligned} \quad (5.61)$$

where in the last step the Bekenstein-Hawking entropy-area formula [95] was used.

Now, as done for the  $D0 - D6$  configuration in the previous Subsection, by exploiting the *first-order formalism* [109] for  $d = 4$  extremal BHs [37, 42] (see also [70] and [74]), one can compute the relevant BH parameters, such as the *ADM mass* (Eq. (3.11)) and the *covariant scalar charges* (Eq. (3.12)), starting from the *fake superpotential*  $\mathcal{W}_{non-BPS,Z \neq 0}$  given by Eq. (5.60). The computations are long but straightforward, and they yield cumbersome results which we thus decide to omit here. We will explicitly analyze some particular configurations in Sect. VI.

However, it is easy to realize that Eq. (5.60) implies the *marginal bound* [107] to be *saturated*, because (see Eq. (3.11))

$$\begin{aligned} & M_{ADM,non-BPS,Z \neq 0}(\mathcal{Z}_\infty, \overline{\mathcal{Z}}_\infty, \Gamma_{D0-D2-D4-D6}) = \\ & = \mathcal{W}_{non-BPS,Z \neq 0}(\mathcal{Z}_\infty, \overline{\mathcal{Z}}_\infty, \Gamma_{D0-D2-D4-D6}) \equiv \\ & \equiv \lim_{\tau \rightarrow 0^-} \mathcal{W}_{non-BPS,Z \neq 0}(\mathcal{Z}(\tau), \overline{\mathcal{Z}}(\tau), \Gamma_{D0-D2-D4-D6}) \end{aligned} \quad (5.62)$$

is equal to the sum of the *ADM masses* of four  $D6$ -branes with appropriate fluxes (for further detail on definition of such brane fluxes, see the related discussion in [59]). Thus, generalizing the related results of [59] and [62], it can be stated that *marginal stability* holds for the most general non-BPS  $Z \neq 0$  attractor flow of the  $\mathcal{N} = 2$ ,  $d = 4$   $st^2$  model.

## VI. ANALYSIS OF PARTICULAR CONFIGURATIONS

In this Section we analyze in depth some particularly simple configurations, generalizing some results in literature [28, 51, 59, 62], but for the much less known case of  $st^2$  model rather than the popular  $stu$  model.

### A. *Magnetic* ( $D0 - D4$ )

The configuration  $D0 - D4$  (also named *magnetic*) of the  $stu$  model has been previously treated in [51] and [59]. In this case of  $st^2$  model, the quantities of the  $U$ -duality transformation (5.3)-(5.5) along  $\mathcal{O}_{non-BPS, Z \neq 0}$  defined by Eqs. (5.42)-(5.46) undergo a major simplification :

$$\lambda = \lambda_0; \quad \varsigma_1 = \varrho_1 = \frac{\sqrt{-\mathcal{I}_4}}{2(p^2)^2} = \sqrt{\frac{-q_0 p^1}{(p^2)^2}}; \quad \varsigma_2 = \varrho_2 = \frac{\sqrt{-\mathcal{I}_4}}{2p^1 p^2} = \sqrt{\frac{-q_0}{p^1}}. \quad (6.1)$$

Correspondingly, the non-BPS  $Z \neq 0$  attractor flow (5.50)-(5.54) acquires the following form (as above, the moduli are here denoted as  $\mathcal{Z}^i \equiv \mathcal{X}^i - i\mathcal{Y}^i$ ;  $i \neq j$  and no sum on repeated  $i$ -indices throughout):

$$\exp[-4U_{non-BPS, Z \neq 0}(\tau)] = h_0(\tau) h_1(\tau) h_2^2(\tau) - b^2; \quad (6.2)$$

$$\mathcal{X}_{non-BPS, Z \neq 0}^1(\tau) = \varsigma_1 \cdot$$

$$\cdot \frac{e^{2\alpha_1} [h_2^2(\tau) + h_0(\tau) h_1(\tau) + 2b] - [h_2^2(\tau) + h_0(\tau) h_1(\tau) - 2b]}{\left\{ \begin{array}{l} e^{2\alpha_1} [h_2^2(\tau) + h_0(\tau) h_1(\tau) + 2b] + 2e^{\alpha_1} [h_2^2(\tau) - h_0(\tau) h_1(\tau)] + \\ + h_2^2(\tau) + h_0(\tau) h_1(\tau) - 2b \end{array} \right\}}; \quad (6.3)$$

$$\mathcal{X}_{non-BPS, Z \neq 0}^2(\tau) = \varsigma_2 \cdot$$

$$\cdot \frac{e^{2\alpha_2} [h_1(\tau) h_2(\tau) + h_0(\tau) h_2(\tau) + 2b] - [h_1(\tau) h_2(\tau) + h_0(\tau) h_2(\tau) - 2b]}{\left\{ \begin{array}{l} e^{2\alpha_2} [h_1(\tau) h_2(\tau) + h_0(\tau) h_2(\tau) + 2b] + 2e^{\alpha_2} [h_1(\tau) h_2(\tau) - h_0(\tau) h_2(\tau)] + \\ + h_1(\tau) h_2(\tau) + h_0(\tau) h_2(\tau) - 2b \end{array} \right\}}; \quad (6.4)$$

$$\mathcal{Y}_{non-BPS,Z \neq 0}^1(\tau) = 4\varsigma_1 \cdot$$

$$\cdot \frac{e^{\alpha_1} \exp[-2U_{non-BPS,Z \neq 0}(\tau)]}{\left\{ \begin{array}{l} e^{2\alpha_1} [h_2^2(\tau) + h_0(\tau) h_1(\tau) + 2b] + 2e^{\alpha_1} [h_2^2(\tau) - h_0(\tau) h_1(\tau)] + \\ + h_2^2(\tau) + h_0(\tau) h_1(\tau) - 2b \end{array} \right\}} \cdot \quad (6.5)$$

$$\mathcal{Y}_{non-BPS,Z \neq 0}^2(\tau) = 4\varsigma_2 \cdot$$

$$\cdot \frac{e^{\alpha_2} \exp[-2U_{non-BPS,Z \neq 0}(\tau)]}{\left\{ \begin{array}{l} e^{2\alpha_2} [h_1(\tau) h_2(\tau) + h_0(\tau) h_2(\tau) + 2b] + 2e^{\alpha_2} [h_1(\tau) h_2(\tau) - h_0(\tau) h_2(\tau)] + \\ + h_1(\tau) h_2(\tau) + h_0(\tau) h_2(\tau) - 2b \end{array} \right\}} \cdot \quad (6.6)$$

It is worth pointing out that the  $D0 - D4$  configuration supports *axion-free* non-BPS  $Z \neq 0$  attractor flow(s); when considering the *near-horizon limit*, and thus the critical, charge-dependent values of the moduli, this is consistent with the analysis performed in [15, 41, 47] for the *stu* case. An *axion-free* attractor flow solution of Eqs. (6.2)-(6.6) can be obtained *e.g.* by putting

$$\alpha_i = 0 \quad \forall i = 1, 2. \quad (6.7)$$

$$b = 0, \quad (6.8)$$

and it reads as follows:

$$\begin{aligned} \exp[-4U_{non-BPS,Z \neq 0, axion-free}(\tau)] &= h_0(\tau) h_1(\tau) h_2^2(\tau); \\ \mathcal{X}_{non-BPS,Z \neq 0, axion-free}^i(\tau) &= 0, \quad i = 1, 2. \\ \mathcal{Y}_{non-BPS,Z \neq 0, axion-free}^1(\tau) &= \varsigma_1 \sqrt{\frac{h_0(\tau) h_1(\tau)}{h_2^2(\tau)}}. \\ \mathcal{Y}_{non-BPS,Z \neq 0, axion-free}^2(\tau) &= \varsigma_2 \sqrt{\frac{h_0(\tau)}{h_1(\tau)}}. \end{aligned} \quad (6.9)$$

Furthermore, within the additional assumption (6.7), Eqs. (6.2)-(6.6) yield the particular solution of the one obtained in [59] for the *stu* case.

Always considering a framework in which the assumption (6.7) holds true, Eqs. (5.60) yields that the non-BPS  $Z \neq 0$  *fake superpotential* in the  $D0 - D4$  configuration has the following expression:

$$\mathcal{W}_{non-BPS, Z \neq 0}|_{\alpha_i=0 \ \forall i}(\mathcal{Z}, \overline{\mathcal{Z}}, \Gamma_{D0-D4}) = e^{K/2} \cdot [-q_0 + p^1 |t|^2 + p^2 (s\bar{t} + t\bar{s})]. \quad (6.10)$$

The *axion-free* version of such a *fake superpotential* reads as follows:

$$\mathcal{W}_{non-BPS, Z \neq 0, axion-free}(\mathcal{Y}, \Gamma_{D0-D4}) = \frac{1}{2\sqrt{2}} \frac{[-q_0 + p^1 (\mathcal{Y}^2)^2 + 2p^2 \mathcal{Y}^1 \mathcal{Y}^2]}{\sqrt{\mathcal{Y}^1} \mathcal{Y}^2}. \quad (6.11)$$

The existence of a *first-order formalism* in the non-BPS  $Z \neq 0$ -supporting (branch of the)  $D0 - D4$  configuration of the  $st^2$  model, based on the *fake superpotential* given by Eq. (6.10), gives a simple explanation of the *integrability* of the equations of motion of scalars, for the  $st^2$  case.

Now, as done above for the  $D0 - D6$  and  $D0 - D2 - D4 - D6$  configurations, by exploiting the *first-order formalism* for  $d = 4$  extremal BHs, one can compute the relevant BH parameters, such as the *ADM mass* and the *covariant scalar charges*, starting from the *fake superpotential*  $\mathcal{W}_{non-BPS, Z \neq 0}|_{\alpha_i=0 \ \forall i}$  given by Eq. (6.10).

Concerning the *ADM mass*, by recalling Eq. (3.11) and using Eq. (6.10) one obtains an explicit expression which, after introducing suitable *dressed charges* (see Eq. (6.14)) and putting (see Eq. (5.25))

$$B^1 = B^2 = B, \quad (6.12)$$

is given by Eq. (4.6) of [59] for the  $stu$  model, which we report here for completeness' sake but for the  $st^2$  model:

$$\begin{aligned} M_{ADM, non-BPS, Z \neq 0}|_{\alpha_i=0 \ \forall i}(\mathcal{Z}_\infty, \overline{\mathcal{Z}}_\infty, \Gamma_{D0-D4}) &= \\ &= \lim_{\tau \rightarrow 0^-} \mathcal{W}_{non-BPS, Z \neq 0}|_{\alpha_i=0 \ \forall i}(\mathcal{Z}(\tau), \overline{\mathcal{Z}}(\tau), \Gamma_{D0-D4}) = \\ &= \frac{1}{2\sqrt{2}} [ |Q_0| + (1 + B^2) (P^1 + 2P^2) ], \end{aligned} \quad (6.13)$$

where the *dressed charges* are defined as follows (no summation on repeated indices; notice the different definitions with respect to the  $D0 - D6$  configuration, whose *dressed charges* are given by Eq. (5.25)):

$$Q_0 \equiv \frac{q_0}{\sqrt{\mathcal{Y}_\infty^1 \mathcal{Y}_\infty^2 \mathcal{Y}_\infty^2}}, \quad P^i \equiv \frac{\sqrt{\mathcal{Y}_\infty^1 \mathcal{Y}_\infty^2 \mathcal{Y}_\infty^2}}{\mathcal{Y}_\infty^i} p^i, \quad i = 1, 2. \quad (6.14)$$

By recalling Eq. (3.12) and using Eq. (6.10), one can compute the *covariant scalar charges* of the non-BPS  $Z \neq 0$  attractor flow in the  $D0 - D4$  configuration. Within the simplifying assumptions (6.7) and (6.12), one obtains the following explicit expressions ( $i \neq j$ , no sum on repeated indices):

$$\begin{aligned} & \Sigma_{\mathcal{X},i,non-BPS,Z \neq 0}(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D0-D4}) \equiv \\ & \equiv \lim_{\tau \rightarrow 0^-} \left( \frac{\partial \mathcal{W}_{non-BPS,Z \neq 0}|_{\alpha_m=0 \ \forall m}}{\partial \mathcal{X}^i} \right) (\mathcal{Z}(\tau), \bar{\mathcal{Z}}(\tau), \Gamma_{D0-D4}); \end{aligned} \quad (6.15)$$

$$\begin{aligned} & \Sigma_{\mathcal{Y},i,non-BPS,Z \neq 0}(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D0-D4}) \equiv \\ & \equiv \lim_{\tau \rightarrow 0^-} \left( \frac{\partial \mathcal{W}_{non-BPS,Z \neq 0}|_{\alpha_m=0 \ \forall m}}{\partial \mathcal{Y}^i} \right) (\mathcal{Z}(\tau), \bar{\mathcal{Z}}(\tau), \Gamma_{D0-D4}), \end{aligned} \quad (6.16)$$

then we can compute,

$$\begin{aligned} & \Sigma_{\mathcal{X},1,non-BPS,Z \neq 0}(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D0-D4}) = \\ & = 2\sqrt{2} \mathcal{X}_\infty P^2 \end{aligned} \quad (6.17)$$

$$\begin{aligned} & \Sigma_{\mathcal{X},2,non-BPS,Z \neq 0}(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D0-D4}) = \\ & = \sqrt{2} \mathcal{X}_\infty (P^1 + P^2) \end{aligned} \quad (6.18)$$

$$\begin{aligned} & \Sigma_{\mathcal{Y},1,non-BPS,Z \neq 0}(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D0-D4}) \\ & = \frac{\mathcal{Y}_\infty}{\sqrt{2}} \left( -|Q_0| - 2P^1 + (1 - B^2)(P^1 + 2P^2) \right) \end{aligned} \quad (6.19)$$

$$\begin{aligned} & \Sigma_{\mathcal{Y},2,non-BPS,Z \neq 0}(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D0-D4}) \\ & = \frac{\mathcal{Y}_\infty}{\sqrt{2}} \left( -|Q_0| - 2P^2 + (1 - B^2)(P^1 + 2P^2) \right) \end{aligned} \quad (6.20)$$

where the split in *axionic scalar charges*  $\Sigma_{\mathcal{X},i}$  and *dilatonic scalar charges*  $\Sigma_{\mathcal{Y},i}$  was performed.

Recalling that  $\mathcal{W}_{\frac{1}{2}-BPS} = |Z|$  in all  $\mathcal{N} = 2$ ,  $d = 4$  supergravities, and using Eqs. (2.8),

(3.11) and (3.12), for the  $\frac{1}{2}$ -BPS attractor flow one obtains the following expressions :

$$\begin{aligned} M_{ADM, \frac{1}{2}-BPS}(\mathcal{Z}_\infty, \bar{\mathcal{Z}}_\infty, \Gamma_{D0-D4}) &= \lim_{\tau \rightarrow 0^-} |Z|(\mathcal{Z}(\tau), \bar{\mathcal{Z}}(\tau), \Gamma_{D0-D4}) = \\ &= \frac{1}{2\sqrt{2}} \sqrt{(1+B^2)^2 (P^1+2P^2)^2 - 2(-1+B^2)Q_0(P^1+2P^2) + Q_0^2}; \end{aligned} \quad (6.21)$$

$$\begin{aligned} \Sigma_{\mathcal{X}, 1, \frac{1}{2}-BPS}(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D0-D4}) &\equiv \lim_{\tau \rightarrow 0^-} \left( \frac{\partial |Z|}{\partial \mathcal{X}} \right) (\mathcal{Z}(\tau), \bar{\mathcal{Z}}(\tau), \Gamma_{D0-D4}) = \\ &= \frac{\mathcal{X}_\infty}{M_{ADM, \frac{1}{2}-BPS}} P^2 [(1+B^2)(P^1+2P^2) - Q_0]; \end{aligned} \quad (6.22)$$

$$\begin{aligned} \Sigma_{\mathcal{X}, 2, \frac{1}{2}-BPS}(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D0-D4}) &\equiv \lim_{\tau \rightarrow 0^-} \left( \frac{\partial |Z|}{\partial \mathcal{X}} \right) (\mathcal{Z}(\tau), \bar{\mathcal{Z}}(\tau), \Gamma_{D0-D4}) = \\ &= \frac{\mathcal{X}_\infty^2}{2M_{ADM, \frac{1}{2}-BPS}} (P^1+P^2) [(1+B^2)(P^1+2P^2) - Q_0]; \end{aligned} \quad (6.23)$$

$$\begin{aligned} \Sigma_{\mathcal{Y}, 1, \frac{1}{2}-BPS}(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D0-D4}) &\equiv \lim_{\tau \rightarrow 0^-} \left( \frac{\partial |Z|}{\partial \mathcal{Y}} \right) (\mathcal{Z}(\tau), \bar{\mathcal{Z}}(\tau), \Gamma_{D0-D4}) = \\ &= -\frac{\mathcal{Y}_\infty}{4M_{ADM, \frac{1}{2}-BPS}} \cdot \\ &\cdot \left[ Q_0^2 + 2(1+B^2)P^1(P^1+2P^2) + 2(P^1-B^2(P^1+2P^2))Q_0 + (B^4-1)(P^1+2P^2)^2 \right]. \end{aligned} \quad (6.24)$$

$$\begin{aligned} \Sigma_{\mathcal{Y}, 2, \frac{1}{2}-BPS}(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D0-D4}) &\equiv \lim_{\tau \rightarrow 0^-} \left( \frac{\partial |Z|}{\partial \mathcal{Y}} \right) (\mathcal{Z}(\tau), \bar{\mathcal{Z}}(\tau), \Gamma_{D0-D4}) = \\ &= -\frac{\mathcal{Y}_\infty}{4M_{ADM, \frac{1}{2}-BPS}} \cdot \\ &\cdot \left[ Q_0^2 + 2(1+B^2)P^2(P^1+2P^2) + 2(P^2-B^2(P^1+2P^2))Q_0 + (B^4-1)(P^1+2P^2)^2 \right]. \end{aligned} \quad (6.25)$$

For what concerns the non-BPS  $Z = 0$  attractor flow in the  $D0 - D4$  configuration, by recalling the treatment of Sect. IV and using Eqs. (3.11) and (3.12), one formally obtains the same expressions (6.21)-(6.25) for the *ADM mass* and *covariant (axionic/dilatonic) scalar charges*, with the only difference of the magnetic charges  $p^1, p^2$  being changed to their absolute values. The definition (5.25) of the *B-field(s)* must change accordingly; for instance, in the case  $p^1 > 0, p^2 < 0$ , the (unique, in the assumption (6.12)) *B-field* must be defined as follows:

$$B \equiv \frac{\mathcal{X}_\infty^1}{\mathcal{Y}_\infty^1} = -\frac{\mathcal{X}_\infty^2}{\mathcal{Y}_\infty^2}, \quad (6.26)$$

where  $\mathcal{X}_\infty^2 < 0$ .

Thence, one can for example compare the *ADM masses*. Taking into account the above results, it makes sense to compare only the *ADM masses* pertaining to the  $\frac{1}{2}$ -BPS and non-BPS  $Z \neq 0$  flows. By introducing the *gap*  $\Delta$  between the squared *ADM masses* as follows:

$$\Delta(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma) \equiv M_{ADM, non-BPS, Z \neq 0}^2(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma) - M_{\frac{1}{2}-BPS}^2(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma), \quad (6.27)$$

and using Eqs. (6.13) and (6.21), one achieves the following result, holding for the  $D0 - D4$  configuration of the  $st^2$  model:

$$\begin{aligned} \Delta(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D0-D4}) &\equiv M_{non-BPS, Z \neq 0}^2|_{\alpha_i=0 \ \forall i}(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D0-D4}) - M_{\frac{1}{2}-BPS}^2(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D0-D4}) = \\ &= \frac{1}{2}B^2|Q_0|(P^1 + 2P^2) \geq 0. \end{aligned} \quad (6.28)$$

This is in a sense the difference generalizing the *BPS bound* [110] to the whole attractor flow (in the non-BPS  $Z \neq 0$ -supporting branch of the *magnetic* charge configuration).

$\Delta$  is dilaton-dependent and strictly positive all along the non-BPS  $Z \neq 0$  attractor flow. At the infinity, by using the *dressed charges* defined by Eq. (6.14), the result given by Eq. (4.8) of [59] is recovered for the particular case of *stu* model. Thus, the *BPS bound* [110] holds not only at the BH event horizon ( $r = r_H$ ), but actually (in a dilaton-dependent way) all along the non-BPS  $Z \neq 0$  attractor flow (*i.e.*  $\forall r \geq r_H$ ).

Of course, by relaxing the simplifying conditions (6.7) and/or (6.12), *i.e.* by considering non-vanishing  $\alpha_i$ s (constrained by Eq. (5.19)) and/or different,  $i$ -indexed *B-fields*, a much richer situation arises, but the main features of the framework, outlined above, are left unchanged.

## B. *Electric* ( $D2 - D6$ )

The configuration  $D2 - D6$  (also named *electric*) of the *stu* model has been previously treated in [28] and [62]. Here we do the same exercise for the less known  $st^2$  model. Analogously to what happens in the  $D0 - D4$  (*magnetic*) configuration, in this case the quantities of the  $U$ -duality transformation (5.3)-(5.5) along  $\mathcal{O}_{non-BPS, Z \neq 0}$  defined by Eqs. (5.42)-(5.46) undergo a major simplification (the prime denotes the charges in the considered configuration):

$$\lambda = -\lambda_0; \quad \varsigma_1 = \varrho_1 = -\sqrt{\frac{q_2'^2}{4p'^0 q_1'}}; \quad \varsigma_2 = \varrho_2 = -\sqrt{\frac{q_1'}{p'^0}}. \quad (6.29)$$

Correspondingly, the non-BPS  $Z \neq 0$  attractor flow (5.50)-(5.54) acquires the following form (as above, the moduli are here denoted as  $\mathcal{Z}^i \equiv \mathcal{X}^i - i\mathcal{Y}^i$ ):

$$\exp[-4U_{non-BPS,Z \neq 0}(\tau)] = h_0(\tau) h_1(\tau) h_2^2(\tau) - b^2; \quad (6.30)$$

$$\mathcal{X}_{non-BPS,Z \neq 0}^1(\tau) = \varsigma_1 \cdot$$

$$\cdot \frac{e^{2\alpha_1} [h_2^2(\tau) + h_0(\tau) h_1(\tau) + 2b] - [h_2^2(\tau) + h_0(\tau) h_1(\tau) - 2b]}{\left\{ \begin{array}{l} e^{2\alpha_1} [h_2^2(\tau) + h_0(\tau) h_1(\tau) + 2b] - 2e^{\alpha_1} [h_2^2(\tau) - h_0(\tau) h_1(\tau)] + \\ + h_2^2(\tau) + h_0(\tau) h_1(\tau) - 2b \end{array} \right\}}; \quad (6.31)$$

$$\mathcal{X}_{non-BPS,Z \neq 0}^2(\tau) = \varsigma_2 \cdot$$

$$\cdot \frac{e^{2\alpha_2} [h_1(\tau) h_2(\tau) + h_0(\tau) h_2(\tau) + 2b] - [h_1(\tau) h_2(\tau) + h_0(\tau) h_2(\tau) - 2b]}{\left\{ \begin{array}{l} e^{2\alpha_2} [h_1(\tau) h_2(\tau) + h_0(\tau) h_2(\tau) + 2b] - 2e^{\alpha_2} [h_1(\tau) h_2(\tau) - h_0(\tau) h_2(\tau)] + \\ + h_1(\tau) h_2(\tau) + h_0(\tau) h_2(\tau) - 2b \end{array} \right\}}; \quad (6.32)$$

$$\mathcal{Y}_{non-BPS,Z \neq 0}^1(\tau) = -4\varsigma_1 \cdot$$

$$\cdot \frac{e^{\alpha_1} \exp[-2U_{non-BPS,Z \neq 0}(\tau)]}{\left\{ \begin{array}{l} e^{2\alpha_1} [h_2^2(\tau) + h_0(\tau) h_1(\tau) + 2b] - 2e^{\alpha_1} [h_2^2(\tau) - h_0(\tau) h_1(\tau)] + \\ + h_2^2(\tau) + h_0(\tau) h_1(\tau) - 2b \end{array} \right\}}. \quad (6.33)$$

$$\mathcal{Y}_{non-BPS,Z \neq 0}^2(\tau) = -4\varsigma_2.$$

$$\frac{e^{\alpha_2} \exp[-2U_{non-BPS,Z \neq 0}(\tau)]}{\left\{ \begin{aligned} &e^{2\alpha_2} [h_1(\tau) h_2(\tau) + h_0(\tau) h_2(\tau) + 2b] - 2e^{\alpha_2} [h_1(\tau) h_2(\tau) - h_0(\tau) h_2(\tau)] + \\ &+ h_1(\tau) h_2(\tau) + h_0(\tau) h_2(\tau) - 2b \end{aligned} \right\}}. \quad (6.34)$$

It is worth pointing out that the  $D2 - D6$  configuration supports *axion-free* non-BPS  $Z \neq 0$  attractor flow(s); when considering the *near-horizon limit*, and thus the critical, charge-dependent values of the moduli, this is consistent with the analysis performed in [15, 41, 47] for the *stu* case. An *axion-free* attractor flow solution of Eqs. (6.30)-(6.34) can be obtained *e.g.* by assuming the conditions given by Eqs. (6.7) and (6.8), and it reads as follows:

$$\exp[-4U_{non-BPS,Z \neq 0, axion-free}(\tau)] = h_0(\tau) h_1(\tau) h_2^2(\tau); \quad (6.35)$$

$$\mathcal{X}_{non-BPS,Z \neq 0, axion-free}^i(\tau) = 0, \quad \forall i = 1, 2.$$

$$\begin{aligned} \mathcal{Y}_{non-BPS,Z \neq 0, axion-free}^1(\tau) &= -\varsigma_1 \sqrt{\frac{h_2^2(\tau)}{h_0(\tau) h_1(\tau)}}; \\ \mathcal{Y}_{non-BPS,Z \neq 0, axion-free}^2(\tau) &= -\varsigma_2 \sqrt{\frac{h_1(\tau)}{h_0(\tau)}}. \end{aligned} \quad (6.36)$$

As done for the *magnetic* configuration, in order to further simplify Eqs. (6.30)-(6.34) and (6.35)-(6.36), one can consider the particular case constrained by Eq. (6.7). Within such an additional assumption, if we perform a similar kind of computation for the *stu* model, the solution obtained in [62], generalizing the one of [28], can be recovered as pointed out in [99].

Furthermore, within the simplifying assumption (6.7), Eq. (5.60) yields that the non-BPS  $Z \neq 0$  *fake superpotential* in the  $D2 - D6$  configuration has the following expression:

$$\mathcal{W}_{non-BPS,Z \neq 0}|_{\alpha_i=0 \ \forall i}(\mathcal{Z}, \overline{\mathcal{Z}}, \Gamma_{D2-D6}) = e^{K/2} |s| |t|^2 \cdot \left[ p'^0 + \frac{q'_1}{|t|^2} + \frac{q'_2}{2} \frac{(s\bar{t} + t\bar{s})}{|s|^2 |t|^2} \right]. \quad (6.37)$$

The *axion-free* version of such a *fake superpotential* (*e.g.* pertaining to the solution (6.35)-(6.36)) reads as follows:

$$\mathcal{W}_{non-BPS,Z \neq 0}|_{\alpha_i=0 \ \forall i, \ axion-free}(\mathcal{Z}, \overline{\mathcal{Z}}, \Gamma_{D2-D6}) = \frac{1}{2\sqrt{2}} \sqrt{\mathcal{Y}^1 \mathcal{Y}^2} \left[ p'^0 + \frac{q'_1}{(\mathcal{Y}^2)^2} + \frac{q'_2}{\mathcal{Y}^1 \mathcal{Y}^2} \right]. \quad (6.38)$$

The existence of a *first-order formalism* in the non-BPS  $Z \neq 0$ -supporting (branch of the)  $D2-D6$  configuration of the  $st^2$  model, based on the *fake superpotential* given by Eq. (6.37), gives an explanation of the *integrability* of the equations of motion of scalars supported by the *electric* configuration (see the treatment of [62] applicable for the *stu* case).

Now, as done above for the  $D0-D6$ ,  $D0-D2-D4-D6$  and  $D0-D4$  configurations, by exploiting the *first-order formalism* for  $d = 4$  extremal BHs, one can compute the relevant BH parameters, such as the *ADM mass* and the *covariant scalar charges*, starting from the *fake superpotential*  $\mathcal{W}_{non-BPS, Z \neq 0}|_{\alpha_i=0 \ \forall i}$  given by Eq. (6.37).

Concerning the *ADM mass*, by recalling Eq. (3.11) and using Eqs. (6.37), (5.25) and (6.12), one obtains an explicit expression which, after introducing suitable *dressed charges* (see Eq. (6.40)), reads as follows:

$$\begin{aligned} M_{ADM, non-BPS, Z \neq 0}|_{\alpha_i=0 \ \forall i}(\mathcal{Z}_\infty, \overline{\mathcal{Z}}_\infty, \Gamma_{D2-D6}) &= \\ = \lim_{\tau \rightarrow 0^-} \mathcal{W}_{non-BPS, Z \neq 0}|_{\alpha_i=0 \ \forall i}(\mathcal{Z}(\tau), \overline{\mathcal{Z}}(\tau), \Gamma_{D2-D6}) &= \\ = \frac{\sqrt{1+B^2}}{2\sqrt{2}} [(1+B^2)P'^0 + (Q'_1 + Q'_2)], \end{aligned} \quad (6.39)$$

where the *dressed charges* are defined as follows (no summation on repeated indices; notice the different definitions with respect to the  $D0-D6$  and  $D0-D4$  configurations, whose *dressed charges* are given by Eqs. (5.25) and (6.14), respectively):

$$P'^0 \equiv p'^0 \sqrt{\mathcal{Y}_\infty^1 \mathcal{Y}_\infty^2 \mathcal{Y}_\infty^2}, \quad Q'_i \equiv \frac{\mathcal{Y}_\infty^i}{\sqrt{\mathcal{Y}_\infty^1 \mathcal{Y}_\infty^2 \mathcal{Y}_\infty^2}} q'_i. \quad (6.40)$$

By recalling Eq. (3.12) and using Eq. (6.37), one can compute the *covariant scalar charges* of the non-BPS  $Z \neq 0$  attractor flow in the  $D2-D6$  configuration. Within the simplifying assumptions (6.7) and (6.12), one obtains the following explicit expressions :

$$\begin{aligned} \Sigma_{\mathcal{X}, i, non-BPS, Z \neq 0}(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D2-D6}) &\equiv \\ \equiv \lim_{\tau \rightarrow 0^-} \left( \frac{\partial \mathcal{W}_{non-BPS, Z \neq 0}|_{\alpha_m=0 \ \forall m}}{\partial \mathcal{X}^i} \right) (\mathcal{Z}(\tau), \overline{\mathcal{Z}}(\tau), \Gamma_{D2-D6}); \end{aligned} \quad (6.41)$$

$$\begin{aligned} \Sigma_{\mathcal{Y}, i, non-BPS, Z \neq 0}(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D2-D6}) &\equiv \\ \equiv \lim_{\tau \rightarrow 0^-} \left( \frac{\partial \mathcal{W}_{non-BPS, Z \neq 0}|_{\alpha_m=0 \ \forall m}}{\partial \mathcal{Y}^i} \right) (\mathcal{Z}(\tau), \overline{\mathcal{Z}}(\tau), \Gamma_{D2-D6}), \end{aligned} \quad (6.42)$$

$$\begin{aligned}
\Sigma_{\mathcal{X},1,non-BPS,Z \neq 0}(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D2-D6}) &= \\
&= \sqrt{2} \frac{\mathcal{X}_\infty^1}{\sqrt{1+B^2}} [(1+B^2)P'^0 + Q'_1]
\end{aligned} \tag{6.43}$$

$$\begin{aligned}
\Sigma_{\mathcal{X},2,non-BPS,Z \neq 0}(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D2-D6}) &= \\
&= \sqrt{2} \frac{\mathcal{X}_\infty^2}{\sqrt{1+B^2}} \left[ (1+B^2)P'^0 + \frac{Q'_2}{2} \right]
\end{aligned} \tag{6.44}$$

$$\begin{aligned}
\Sigma_{\mathcal{Y},1,non-BPS,Z \neq 0}(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D2-D6}) &= \\
&= \frac{\mathcal{Y}_\infty^1}{\sqrt{2} \sqrt{1+B^2}} \left[ (B^4-1)P'^0 - 2Q'_1 + (1+B^2)^2(Q'_1 + Q'_2) \right]
\end{aligned} \tag{6.45}$$

$$\begin{aligned}
\Sigma_{\mathcal{Y},2,non-BPS,Z \neq 0}(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D2-D6}) &= \\
&= \frac{\mathcal{Y}_\infty^2}{\sqrt{2} \sqrt{1+B^2}} \left[ (B^4-1)P'^0 - Q'_2 + (1+B^2)^2(Q'_1 + Q'_2) \right]
\end{aligned} \tag{6.46}$$

where, as for the  $D0-D4$  configuration, the split in *axionic scalar charges*  $\Sigma_{\mathcal{X},i}$  and *dilatonic scalar charges*  $\Sigma_{\mathcal{Y},i}$  was performed.

Recalling once again that  $\mathcal{W}_{\frac{1}{2}-BPS} = |Z|$  in all  $\mathcal{N} = 2$ ,  $d = 4$  supergravities, and using Eqs. (2.8), (3.11) and (3.12), for the  $\frac{1}{2}$ -BPS attractor flow one obtains the following expressions :

$$\begin{aligned}
M_{ADM,\frac{1}{2}-BPS}(\mathcal{Z}_\infty, \overline{\mathcal{Z}}_\infty, \Gamma_{D2-D6}) &= \\
&= \lim_{\tau \rightarrow 0^-} |Z|(\mathcal{Z}(\tau), \overline{\mathcal{Z}}(\tau), \Gamma_{D2-D6}) = \\
&= \frac{\sqrt{1+B^2}}{2\sqrt{2}} \sqrt{(1+B^2)^2(P'^0)^2 - 2(-1+B^2)|P'^0|(Q'_1 + Q'_2) + (Q'_1 + Q'_2)^2};
\end{aligned} \tag{6.47}$$

$$\begin{aligned}
\Sigma_{\mathcal{X},1,\frac{1}{2}-BPS}(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D2-D6}) &\equiv \lim_{\tau \rightarrow 0^-} \left( \frac{\partial |Z|}{\partial \mathcal{X}^1} \right) (\mathcal{Z}(\tau), \overline{\mathcal{Z}}(\tau), \Gamma_{D2-D6}) = \\
&= \frac{\mathcal{X}_\infty^1}{2M_{ADM,\frac{1}{2}-BPS}} \cdot \\
&\cdot \left\{ (1+B^2)^2(P'^0)^2 + Q'_1(Q'_1 + Q'_2) + |P'^0| [(-3+B^2)Q'_1 + (1+B^2)(Q'_1 + Q'_2)] \right\};
\end{aligned} \tag{6.48}$$

$$\begin{aligned}
\Sigma_{\mathcal{X}, 2, \frac{1}{2}-BPS}(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D2-D6}) &\equiv \lim_{\tau \rightarrow 0^-} \left( \frac{\partial |Z|}{\partial \mathcal{X}^2} \right) (\mathcal{Z}(\tau), \overline{\mathcal{Z}}(\tau), \Gamma_{D2-D6}) = \\
&= \frac{\mathcal{X}_\infty^2}{2M_{ADM, \frac{1}{2}-BPS}} \cdot \\
&\cdot \left\{ (1+B^2)^2 (P'^0)^2 + \frac{Q'_2(Q'_1+Q'_2)}{2} + |P'^0| \left[ \frac{(-3+B^2)Q'_2}{2} + (1+B^2)(Q'_1+Q'_2) \right] \right\};
\end{aligned} \tag{6.49}$$

$$\begin{aligned}
\Sigma_{\mathcal{Y}, 1, \frac{1}{2}-BPS}(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D2-D6}) &\equiv \lim_{\tau \rightarrow 0^-} \left( \frac{\partial |Z|}{\partial \mathcal{Y}^1} \right) (\mathcal{Z}(\tau), \overline{\mathcal{Z}}(\tau), \Gamma_{D2-D6}) = \\
&= -\frac{\mathcal{Y}_\infty^1}{4M_{ADM, \frac{1}{2}-BPS}} \cdot \\
&\cdot \left\{ \begin{aligned} &(-1+B^2)(1+B^2)^2 (P'^0)^2 + (1+B^2)(Q'_1+Q'_2)^2 + \\ &+ 2Q'_1 [(-1+3B^2)|P'^0| - (Q'_1+Q'_2)] - 2B^2(1+B^2)|P'^0|(Q'_1+Q'_2) \end{aligned} \right\}.
\end{aligned} \tag{6.50}$$

$$\begin{aligned}
\Sigma_{\mathcal{Y}, 2, \frac{1}{2}-BPS}(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D2-D6}) &\equiv \lim_{\tau \rightarrow 0^-} \left( \frac{\partial |Z|}{\partial \mathcal{Y}^2} \right) (\mathcal{Z}(\tau), \overline{\mathcal{Z}}(\tau), \Gamma_{D2-D6}) = \\
&= -\frac{\mathcal{Y}_\infty^2}{4M_{ADM, \frac{1}{2}-BPS}} \cdot \\
&\cdot \left\{ \begin{aligned} &(-1+B^2)(1+B^2)^2 (P'^0)^2 + (1+B^2)(Q'_1+Q'_2)^2 + \\ &+ Q'_2 [(-1+3B^2)|P'^0| - (Q'_1+Q'_2)] - 2B^2(1+B^2)|P'^0|(Q'_1+Q'_2) \end{aligned} \right\}.
\end{aligned} \tag{6.51}$$

Hence, one can, as done for the *magnetic* configuration in Subsect. VIA, also for the *electric* configuration, compute the difference between the squared non-BPS  $Z \neq 0$  *fake superpotential* and the squared absolute value of the  $\mathcal{N} = 2, d = 4$  central charge along the non-BPS  $Z \neq 0$  attractor flow and compare the *ADM masses*. After all dusts get settled, one achieves the following result, holding for the  $D2 - D6$  configuration of the  $st^2$  model:

$$\begin{aligned}
&\Delta(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D2-D6}) \equiv \\
&\equiv M_{non-BPS, Z \neq 0}^2|_{\alpha_i=0 \forall i}(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D2-D6}) - M_{ADM, \frac{1}{2}-BPS}(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D2-D6}) = \\
&= \frac{1}{2} (1+B^2) B^2 |P'^0| (Q'_1+Q'_2) \geq 0.
\end{aligned} \tag{6.52}$$

Unlike what happens for the *magnetic* configuration, for *electric* configuration  $\Delta$  does depend also on axions, nevertheless it is still strictly positive all along the non-BPS  $Z \neq 0$  attractor flow. At infinity, by using the *dressed charges* defined by Eq. (6.40), the following result is achieved:

$$\Delta(\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma) = \frac{P'^0}{2}(1 + B^2)(Q'_1 + Q'_2). \quad (6.53)$$

Thus, the *BPS bound* [110] holds not only at the BH event horizon ( $r = r_H$ ), but actually (in a scalar-dependent way) all along the non-BPS  $Z \neq 0$  attractor flow (*i.e.*  $\forall r \geq r_H$ ).

Of course, by relaxing the simplifying conditions (6.7) and/or (6.12), *i.e.* by considering non-vanishing  $\alpha_i$ s (constrained by Eq. (5.19)) and/or different,  $i$ -indexed  $B$ -fields, a much richer situation arises, but the main features of the framework, outlined above, are left unchanged.

By noticing that the  $D0 - D4$  (*magnetic*) and  $D2 - D6$  (*electric*) configurations are reciprocally *dual* in  $d = 4$  and recalling the treatment of Subsect. V A 1, it is worth computing the matrices  $M_{i,D0-D4 \rightarrow D2-D6}$  representing the  $U$ -duality transformation along the charge orbit  $\mathcal{O}_{non-BPS, Z \neq 0}$  which connects (the non-BPS  $Z \neq 0$ -supporting *branches* of) such two charge configurations. In order to do this, we exploit the treatment given in Subsect. V A 1, by performing the following steps:

$$\begin{matrix} (q_0, p^i) \\ D0 - D4 \end{matrix} \longrightarrow \begin{matrix} (q, p) \\ D0 - D6 \end{matrix} \longrightarrow \begin{matrix} (q'_i, p'^0) \\ D2 - D6 \end{matrix}. \quad (6.54)$$

For the step  $D0 - D4 \rightarrow D0 - D6$ , we consider  $M_i^{-1}$  given by Eq. (5.41), along with the definitions (5.42)-(5.46) specified for the configuration  $D0 - D4$ , obtaining  $M_{i,D0-D4 \rightarrow D0-D6}^{-1}$ . Then, for the the step  $D0 - D6 \rightarrow D2 - D6$ , we take  $M_i$  given by Eq. (5.41), along with the definitions (5.42)-(5.46) specified for the configuration  $D2 - D6$ , obtaining  $M_{i,D0-D6 \rightarrow D2-D6}$ . Thus (also recall Eq. (5.3)):

$$\begin{aligned} (M_{1,D0-D4 \rightarrow D2-D6})_{a'}^{b'} &= (M_{1,D0-D6 \rightarrow D2-D6})_{a'}^a (M_{1,D0-D4 \rightarrow D0-D6})_a^{b'} = \\ &= \begin{pmatrix} 0 & -\left(-\frac{q_0 p^1 q_2'^2}{4 p'^0 q_1' (p^2)^2}\right)^{1/4} \\ \left(-\frac{4 p'^0 q_1' (p^2)^2}{q_0 p^1 q_2'^2}\right)^{1/4} & 0 \end{pmatrix}. \end{aligned} \quad (6.55)$$

$$\begin{aligned}
(M_{2,D0-D4 \rightarrow D2-D6})_{a'}^{b'} &= (M_{i,D0-D6 \rightarrow D2-D6})_{a'}^a (M_{2,D0-D4 \rightarrow D0-D6})_a^{b'} = \\
&= \begin{pmatrix} 0 & -\left(-\frac{q_0 q'_1}{p'^0 p^1}\right)^{1/4} \\ \left(-\frac{p'^0 p^1}{q_0 q'_1}\right)^{1/4} & 0 \end{pmatrix}.
\end{aligned} \tag{6.56}$$

Consequently, by recalling Eqs. (5.5) and ((5.47) - (5.48)) one can directly relate the non-BPS  $Z \neq 0$  attractor flows  $\mathcal{Z}_{non-BPS,Z \neq 0,D2-D6}^i(\tau)$  and  $\mathcal{Z}_{non-BPS,Z \neq 0,D0-D4}^i(\tau)$  (respectively given by Eqs. (6.30)-(6.34) and (6.2)-(6.6); recall that  $\mathcal{Z}^i(\tau) = \mathcal{X}^i(\tau) - i\mathcal{Y}^i(\tau)$ ) by the following expression, explicitly showing the *duality* between the  $D0 - D4$  (*magnetic*) and  $D2 - D6$  (*electric*) configurations in  $d = 4$ :

$$\mathcal{Z}_{non-BPS,Z \neq 0,D2-D6}^1(\tau) = -\sqrt{\frac{|q_0| p^1 (q'_2)^2}{4p'^0 q'_1 (p^2)^2}} \frac{1}{\mathcal{Z}_{non-BPS,Z \neq 0,D0-D4}^1(\tau)}. \tag{6.57}$$

$$\mathcal{Z}_{non-BPS,Z \neq 0,D2-D6}^2(\tau) = -\sqrt{\frac{|q_0| q'_1}{p'^0 p^1}} \frac{1}{\mathcal{Z}_{non-BPS,Z \neq 0,D0-D4}^2(\tau)}. \tag{6.58}$$

### C. $D0 - D2 - D4$

The configuration  $D0 - D2 - D4$  of the *stu* model has been previously treated in [62], provoking us to do the same analysis for the less known case of  $st^2$  model. In this case, the quantities of the  $U$ -duality transformation (5.3)-(5.5) along  $\mathcal{O}_{non-BPS,Z \neq 0}$  defined by Eqs. (5.42)-(5.46) have the following form :

$$\begin{aligned}
\lambda &= \lambda_0; \\
\varsigma_1 &= \frac{\sqrt{-\mathcal{I}_4} - p^1 q_1 + p^2 q_2}{2(p^2)^2}; \quad \varrho_1 = \frac{\sqrt{-\mathcal{I}_4} + p^1 q_1 - p^2 q_2}{2(p^2)^2} \\
\varsigma_2 &= \frac{\sqrt{-\mathcal{I}_4} + p^1 q_1}{2p^1 p^2}; \quad \varrho_2 = \frac{\sqrt{-\mathcal{I}_4} - p^1 q_1}{2p^1 p^2}.
\end{aligned} \tag{6.59}$$

Within the additional assumption (6.7) (considered for simplicity' sake), the non-BPS  $Z \neq 0$  attractor flow (5.50)-(5.54) correspondingly acquires the following form (as above, the moduli are here denoted as  $\mathcal{Z}^i \equiv \mathcal{X}^i - i\mathcal{Y}^i$ ):

$$\exp[-4U_{non-BPS,Z \neq 0}(\tau)] = h_0(\tau) h_1(\tau) h_2^2(\tau) - b^2; \quad (6.60)$$

$$\mathcal{X}_{non-BPS,Z \neq 0}^1(\tau) = \frac{\sqrt{-\mathcal{I}_4}}{2(p^2)^2} \frac{b}{h_2^2(\tau)} + \frac{p^2 q_2 - p^1 q_1}{2(p^2)^2}; \quad (6.61)$$

$$\mathcal{X}_{non-BPS,Z \neq 0}^2(\tau) = \frac{\sqrt{-\mathcal{I}_4}}{2p^1 p^2} \frac{b}{h_1(\tau) h_2(\tau)} + \frac{p^1 q_1 - p^2 q_2}{2p^1 p^2}; \quad (6.62)$$

$$\mathcal{Y}_{non-BPS,Z \neq 0}^1(\tau) = \frac{\sqrt{-\mathcal{I}_4}}{2(p^2)^2} \frac{\exp[-2U_{non-BPS,Z \neq 0}(\tau)]}{h_2^2(\tau)}; \quad (6.63)$$

$$\mathcal{Y}_{non-BPS,Z \neq 0}^2(\tau) = \frac{\sqrt{-\mathcal{I}_4}}{2p^1 p^2} \frac{\exp[-2U_{non-BPS,Z \neq 0}(\tau)]}{h_1(\tau) h_2(\tau)}. \quad (6.64)$$

It is worth pointing out that, as the general case  $D0-D2-D4-D6$  (see Subsect. VB 2), the  $D0-D2-D4$  configuration does *not* support *axion-free* non-BPS  $Z \neq 0$  attractor flow(s); when considering the *near-horizon limit*, and thus the critical, charge-dependent values of the moduli, this is consistent with the analysis performed in [15, 41, 47] for the *stu* model.

Furthermore, always within the simplifying assumption (6.7), Eq. (5.60) yields that the non-BPS  $Z \neq 0$  *fake superpotential* in the  $D0-D2-D4$  configuration has the following expression:

$$\begin{aligned} & \mathcal{W}_{non-BPS,Z \neq 0}|_{\alpha_i=0 \ \forall i}(\mathcal{Z}, \bar{\mathcal{Z}}, \Gamma_{D0-D2-D4}) = \\ & = e^{K/2} \left[ -q_0 - \frac{q_1}{2}(s + \bar{s}) - \frac{q_2}{2}(t + \bar{t}) + p^1 |t|^2 + p^2 (s\bar{t} + t\bar{s}) \right]. \end{aligned} \quad (6.65)$$

The existence of a *first-order formalism* in the non-BPS  $Z \neq 0$ -supporting (branch of the)  $D0-D2-D4$  configuration of the  $st^2$  model, based on the *fake superpotential* given by Eq. (6.65), gives an explanation of the *integrability* of the equations of motion of scalars supported by such a configuration (see the treatment of [62] applicable for the *stu* case).

Now, by exploiting the *first-order formalism* for  $d = 4$  extremal BHs, one can compute the relevant BH parameters, such as the *ADM mass* and the *covariant scalar charges*, starting from the *fake superpotential*  $\mathcal{W}_{non-BPS,Z \neq 0}|_{\alpha_i=0 \ \forall i}$  given by Eq. (6.65).

Concerning the *ADM mass*, by recalling Eq. (3.11) and using Eq. (6.65) one obtains the following result:

$$\begin{aligned}
& M_{ADM, non-BPS, Z \neq 0} \big|_{\alpha_i=0 \ \forall i} (\mathcal{Z}_\infty, \overline{\mathcal{Z}}_\infty, \Gamma_{D0-D2-D4}) = \\
& = \lim_{\tau \rightarrow 0^-} \mathcal{W}_{non-BPS, Z \neq 0} \big|_{\alpha_i=0 \ \forall i} (\mathcal{Z}(\tau), \overline{\mathcal{Z}}(\tau), \Gamma_{D0-D2-D4}) = \\
& = \frac{1}{2\sqrt{2}} [|Q_0| - (Q_1 B_1 + Q_2 B_2) + (P^1 + 2P^2) + (P^1 B_2^2 + 2P^2 B_1 B_2)], \tag{6.66}
\end{aligned}$$

where the *dressed charges* are defined by Eqs. (6.14) and (6.40).

By recalling Eq. (3.12) and using Eq. (6.65), one can compute the *covariant scalar charges* of the non-BPS  $Z \neq 0$  attractor flow in the  $D0 - D2 - D4$  configuration within the assumption (6.7), and then one obtains the following explicit expressions :

$$\begin{aligned}
& \Sigma_{\mathcal{X}, i, non-BPS, Z \neq 0} (\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D0-D2-D4}) \equiv \\
& \equiv \lim_{\tau \rightarrow 0^-} \left( \frac{\partial \mathcal{W}_{non-BPS, Z \neq 0} \big|_{\alpha_m=0 \ \forall m}}{\partial \mathcal{X}^i} \right) (\mathcal{Z}(\tau), \overline{\mathcal{Z}}(\tau), \Gamma_{D0-D2-D4}); \tag{6.67}
\end{aligned}$$

$$\begin{aligned}
& \Sigma_{\mathcal{Y}, i, non-BPS, Z \neq 0} (\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D0-D2-D4}) \equiv \\
& \equiv \lim_{\tau \rightarrow 0^-} \left( \frac{\partial \mathcal{W}_{non-BPS, Z \neq 0} \big|_{\alpha_m=0 \ \forall m}}{\partial \mathcal{Y}^i} \right) (\mathcal{Z}(\tau), \overline{\mathcal{Z}}(\tau), \Gamma_{D0-D2-D4}), \tag{6.68}
\end{aligned}$$

and we can write

$$\begin{aligned}
& \Sigma_{\mathcal{X}, 1, non-BPS, Z \neq 0} (\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D0-D2-D4}) = \\
& = \sqrt{2} \mathcal{Y}_\infty^1 (2P^2 B_2 - Q_1) \tag{6.69}
\end{aligned}$$

$$\begin{aligned}
& \Sigma_{\mathcal{X}, 2, non-BPS, Z \neq 0} (\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D0-D2-D4}) = \\
& = \sqrt{2} \mathcal{Y}_\infty^2 (P^1 B_2 + P^2 B_1 - Q_2) \tag{6.70}
\end{aligned}$$

$$\begin{aligned}
& \Sigma_{\mathcal{Y}, 1, non-BPS, Z \neq 0} (\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D0-D2-D4}) = \\
& = \frac{\mathcal{Y}_\infty^1}{\sqrt{2}} [-|Q_0| - 2P^1 + (Q_1 B_1 + Q_2 B_2) + (P^1 + 2P^2) - (P^1 B_2^2 + 2P^2 B_1 B_2)] \tag{6.71}
\end{aligned}$$

$$\begin{aligned}
& \Sigma_{\mathcal{Y}, 2, non-BPS, Z \neq 0} (\mathcal{X}_\infty, \mathcal{Y}_\infty, \Gamma_{D0-D2-D4}) = \\
& = \frac{\mathcal{Y}_\infty^2}{\sqrt{2}} [-|Q_0| - 2P^2 + (Q_1 B_1 + Q_2 B_2) + (P^1 + 2P^2) - (P^1 B_2^2 + 2P^2 B_1 B_2)] \tag{6.72}
\end{aligned}$$

where, as above, the split in *axionic scalar charges*  $\Sigma_{\mathcal{X},i}$  and *dilatonic scalar charges*  $\Sigma_{\mathcal{Y},i}$  was performed, and the definition (5.25) of *B-fields* was used.

As done for the *magnetic* and *electric* configurations (respectively in Subsects. VIA and VIB), also for  $D0 - D2 - D4$  configuration the difference between the squared non-BPS  $Z \neq 0$  *fake superpotential* and the squared absolute value of the  $\mathcal{N} = 2$ ,  $d = 4$  central charge along the non-BPS  $Z \neq 0$  attractor flow can be computed giving the result that  $\Delta$  is strictly positive all along the non-BPS  $Z \neq 0$  attractor flow, due to the fact that  $\mathcal{I}_4$  is strictly negative.

Thus, the *BPS bound* [110] is found to hold not only at the BH event horizon ( $r = r_H$ ), but actually (in a scalar-dependent way) all along the non-BPS  $Z \neq 0$  attractor flow (*i.e.*  $\forall r \geq r_H$ ).

It is here worth pointing out that, by exploiting the procedure outlined in Sect. VI, the results for  $\Delta$  computation can be related one to the others by performing suitable *U*-duality transformations. In such a way, one can also compute  $\Delta$  for the non-BPS  $Z \neq 0$ -supporting branch of the most general (*i.e.*  $D0 - D2 - D4 - D6$ ) BH charge configuration.

## VII. CONCLUSION

In the present paper the analysis and solution of the equations of motion of the scalar fields of the so-called  $st^2$  model [50], consisting of  $\mathcal{N} = 2$ ,  $d = 4$  *ungauged* supergravity coupled to 2 Abelian vector multiplets whose complex scalars span the special Kähler manifold  $\frac{G}{H} = \left( \frac{SU(1,1)}{U(1)} \right)^2$ , has been performed in full detail, filling the gap in the so far, existing supergravity black hole literature. The obtained results complete a so far unknown analysis of the 3 classes of *non-degenerate* attractor flows of the  $st^2$  model in its full generality. It is to be noted that all their features have been studied and compared in this report.

Various comments, remarks, ideas for further developments along the lines of research considered in the present paper are listed below.

- Since the  $st^2$  model is a consistent truncation of the much known *stu* model, given any classical solution in the  $st^2$  model, it can be regarded as a classical solution in the *stu* model if we limit ourselves to a special class of solution for which the two moduli are equal *i.e.*  $u = t$  and moreover, as this is the case, we should be able to derive all

properties of the classical solutions in the  $st^2$  model using the corresponding properties of the  $stu$  model and then choosing a special subset of solutions for which  $u = t$ . This crucial fact has been cross-checked against all the results obtained in this paper and matched with what has been obtained from the corresponding results in [59, 99], after a degeneracy choice of  $u = t$  is made.

- By exploiting the approach considered in Sect. 5 of [50], the  $st^2$  model can be consistently related to the so-called  $stu$  and  $t^3$  models, respectively with 3 and 1 *complex* scalars. Such a procedure enables one to extend all the results obtained for the  $st^2$  case to other SUGRA models. (For results holding true for  $stu$  case, see [99]). Furthermore, by performing the *near-horizon* (*i.e.*  $\tau \rightarrow -\infty$ ) *limit* on the attractor flow solutions, the corresponding attractor solution at the event horizon of the extremal BH can be obtained. This is particularly relevant for the non-BPS  $Z \neq 0$  horizon attractor solutions, hitherto analytically known (in a rather intricate form) only for the  $t^3$  model, so far the only  $\mathcal{N} = 2$ ,  $d = 4$  supergravity model based on *cubic* special Kähler geometry whose Attractor Eqs. had been completely solved. In the *near-horizon limit*, the results of the present paper yield the non-BPS  $Z \neq 0$  horizon attractor flow solutions for the  $st^2$  model.
- The  $st^2$  model has been recently shown to be relevant for the Special entangled Quantum systems, Freudenthal construction and the study of the group of stochastic local operations and classical communications (SLOCC) [97] with one distinguished *qubit* and two bosonic *qubits* in *quantum information theory* and extremal stringy BHs [3] (see also [111] for further recent developments). In the seventh of Refs. [3] the relation between quantum information theory and the theory of extremal stringy BHs was studied within the  $stu$  model, showing that the *three-qubit interpretation* of supersymmetric,  $\frac{1}{2}$ -BPS attractors can be extended also to include non-supersymmetric, non-BPS (both  $Z \neq 0$  and  $Z = 0$ ) ones, performing a classification of the attractor solutions based on the charge codes of *quantum error correction*. However, only *double-extremal* solutions, with *constant, non-dynamical* scalars *all along* the attractor flow, were discussed therein. Thus, as also observed in [62], it would be interesting to extend the analysis of the seventh of Refs. [3] using the full general non-BPS (both  $Z \neq 0$  and  $Z = 0$ ) attractor flow solutions obtained in the present paper and see

whether  $st^2$  model does also find its applicability in the *quantum information theory*.

- The existence of a *first-order formalism* for the equations of motion for the scalar fields (also named *attractor flow Eqs.*) in the background of an extremal BH [37, 42] *in principle* implies the *integrability* of such equations, regardless of their final form. It is particularly relevant for the non-BPS  $Z \neq 0$  attractor flow, as pointed out in Subsects. VI A-VI C. It would be interesting to study the integrability of the equations of motion of the scalars in presence of *quantum* (*perturbative* and/or *non-perturbative*) corrections to the considered  $st^2$  model. For instance, it would be interesting to study the attractor flow Eqs. for a quantum corrected prepotential  $f = st^2 + i\lambda$  [112], with  $\lambda \in \mathbb{R}$ , which is the only correction which preserves the *Peccei-Quinn axion shift symmetry* and modifies the geometry of the scalar manifold (see [58] and Refs. therein). A tempting idea, inspired by the intriguing connection between quantum information theory and extremal BHs mentioned at the previous point, is to consider the quantum, axion-shift-consistent parameter  $\lambda$  as related to the *quantum noise* of the system (see *e.g.* [113] and Refs. therein).
- As found in [108], also observed in [59] for the  $stu$  model and noticed in Sect. III, for the  $st^2$  case an immediate consequence of the general form of  $\frac{1}{2}$ -BPS attractor flow given by Eq. (3.1) is that  $\Gamma_\infty$  for the  $st^2$  model satisfies the  $\frac{1}{2}$ -BPS *Attractor Eqs.* [103]. This determines a sort of “*Attractor Mechanism at spatial infinity*”, mapping the 4 *real* moduli  $(x^1, x^2, y^1, y^2)$  into the 6 *real* constants  $(p_\infty^1, p_\infty^2, q_{1,\infty}, q_{2,\infty})$ , arranged as  $\Gamma_\infty$  and constrained by the 2 *real* conditions (3.5). As noticed in [59], for the  $stu$  model, the absence of *flat* directions in the  $\frac{1}{2}$ -BPS attractor flow for the  $st^2$  model (which is a general feature of  $\mathcal{N} = 2$ ,  $d = 4$  *ungauged* supergravity coupled to Abelian vector multiplets, *at least* as far as the metric of the scalar manifold is strictly positive-definite  $\forall \tau \in \mathbb{R}^-$  [10]) is crucial for the validity of the expression (3.1). As pointed out in Sect. IV, the same holds for the non-BPS  $Z = 0$  case. Indeed, a consequence of the general form of non-BPS  $Z = 0$  attractor flow given by Eq. (4.4) is that  $\Gamma_\infty$  satisfies the *non-BPS  $Z = 0$  Attractor Eqs.* (see *e.g.* [23] and [32]), determining a sort of “*Attractor Mechanism at spatial infinity*”. Analogously to what happens in the  $\frac{1}{2}$ -BPS case, the absence of *flat* directions in the non-BPS  $Z = 0$  attractor flow (which is *not* a general feature of  $\mathcal{N} = 2$ ,  $d = 4$  *ungauged* supergravity coupled to

Abelian vector multiplets, but however holds for the  $st^2$  model [40, 43]) is crucial for the validity of the expression (4.4). Bearing in mind the crucial differences among the non-BPS  $Z \neq 0$  attractor flow and the  $\frac{1}{2}$ -BPS and non-BPS  $Z = 0$  attractor flows, (such as the presence of a 1-dim. real *moduli space*  $SO(1,1)$  *all along* the non-BPS  $Z \neq 0$  attractor flow), it would be interesting to investigate whether there exists any non-BPS  $Z \neq 0$  “*Attractor Mechanism at spatial infinity*”.

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